Mathematics

Class Nine

NATIONAL CURRICULUM AND TEXTBOOK BOARD, BANGLADESH

নারী ও কন্যাশিশুদের শিক্ষা প্রসারের স্বীকৃতি হিসেবে ২০১৪ সালে
ইউনেস্কো 'শান্তিবৃক্ষ' (Peace Tree) পুরস্কার গ্রহণ।

নারী ও কন্যাশিশুদের শিক্ষা প্রসারে বিশেষ অবদান রাখার জন্য ২০১৪ সালে ইউনেস্কো বাংলাদেশের প্রধানমন্ত্রী শেখ হাসিনাকে 'শান্তিবৃক্ষ' (Peace Tree) পুরস্কারে ভূষিত করে। মাননীয় প্রধানমন্ত্রীর হাতে 'শান্তিবৃক্ষ' (Peace Tree) পুরস্কার তুলে দেওয়ার সময় শেখ হাসিনাকে 'সাহসী নারী' হিসেবে অভিহিত করেন ইউনেস্কোর মহাপরিচালক। তিনি বলেন, নারী ও কন্যাশিশুদের ক্ষমতায়নে বাংলাদেশের প্রধানমন্ত্রী বিশ্বমঞ্চের জোরালো এক কণ্ঠ।

কন্যাশিশু ও নারী শিক্ষা প্রসারের ফলে নারীরা সামাজিক, মানসিক ও অর্থনৈতিকভাবে স্বাবলম্বী হচ্ছে, টেকসই উন্নয়নের ভিত্তি রচিত হয়েছে এবং নারীরা নতুন নতুন পেশায় যুক্ত হওয়ার সুযোগ পাচ্ছে।

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Mathematics

Class Nine

(Experimental Edition)

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PREFACE

In this ever-changing world, the concept of life and livelihood is changing every moment. This process of change has been accelerated due to the advancement of technology. There is no alternative to adapting to this fast changing world as technology is changing rapidly ever than before. In the era of fourth industrial revolution, the advancement of artificial intelligence has brought about drastic changes in our employment and lifestyles that will make the relationship among people more and more intimate. Various employment opportunities will be created in near future which we cannot even predict at this moment. We need to take preparation right now so that we can adapt ourselves to that coming future.

Although a huge economic development has taken place throughout the world, problems like climate change, air pollution, migrations and ethnic violence have become much more intense nowadays. The breakouts of pandemics like COVID 19 have crippled the normal lifestyle and economic growth of the world. Thus, different challenges as well as opportunities, have been added to our daily life.

Standing amid the array of challenges and potentials, sustainable and effective solutions are required to transform our large population into a resource. It entails global citizens with knowledge, skill, values, vision, positive attitude, sensitivity, adaptability, humanism and patriotism. Amidst all these, Bangladesh has graduated into a developing nation from the underdeveloped periphery and is continuously trying to achieve the desired goals in order to become a developed country by 2041. Education is one of the most crucial instruments to attain the goals. Hence, there is no alternative to the transformation of our education system. This transformation calls for developing an effective and updated curriculum.

Developing and updating the curriculum is a routine and important activity of National Curriculum and Textbook Board. The curriculum was last revised in 2012. Since then, more than a decade has elapsed. Therefore, there was a need for curriculum revision and development. With this view, various research and technical studies were conducted under NCTB from 2017 to 2019 to analyze the current state of education and identify the learning needs. Based on the researches and technical studies, a competency-based and seamless curriculum from K−12 has been developed to create a competent generation capable of surviving in the new world situation.

Under the framework of this competency based curriculum, the textbooks have been prepared for all streams (General, Madrasah and Vocational) of learners for Class Nine. The authentic experience-driven contents of this textbook were developed with a view to making learning comprehensible and enjoyable. This will connect the textbooks with various life related phenomenon and events that are constantly taking place around us. It is expected that, through this, learning will be much more insightful and lifelong.

In developing the textbooks, due importance has been given to all − irrespective of gender, ethnicity, religion and caste while the needs of the disadvantaged and special children are taken into special considerations.

I would like to thank all who have put their best efforts in writing, editing, revising, illustrating and publishing the textbook.

If any errors or inconsistencies in this experimental version are found or if there is any suggestions for further improvement of this textbook, you are requested to let us know.

Professor Md. Farhadul Islam
Chairman
National Curriculum and Textbook Board, Bangladesh

Dear Students

Welcome to the ninth grade students of secondary level. National Curriculum and Textbook Board, Bangladesh has planned to develop new textbooks for all secondary level students. In continuation of this, the new mathematics book of class 9 has been written. Fundamental changes have been made in the book's presentation, decoration, topics and teaching-learning methodology of mathematics. Surely you have various curiosity about these changes and innovations in class 9 books.

In experiential learning method it is very im-portant to link the subject matter with real life ex-perience. In this context two aspects have been giv-en utmost importance while preparing the book suitable for students class IX. First, they will have the opportunity to solve mathematical problems through hands-on work by observing the objects and events of the familiar environment around them. Second, they can acquire the techniques of using mathematical skills in various tasks of daily life.

A total of nine learning experiences are planned in this book of class 9 . You will participate in these experiences by mathematically analyzing and solv-ing real-life problems. Each learning experience is presented step by step so that you can master mathematical concepts and skills through active participation and use of real materials. This jour-ney of learning mathematics through mathematical inquiry will be as enjoyable for you as you will dis-cover for yourself the relation of mathematical concepts to real life.

The teacher will give you full support in all activi-ties inside and outside the classroom. We also hope that you will be supportive of each other as you participate in the various activities of this learning program and explore different topics of mathemat-ics with your classmates. You will always remem-ber that when all of you have a cooperative spirit you can do any task successfully. We hope this book will play an important role in ensuring an effective and enjoyable learning journey for you in the world of mathematics. The textbook will serve as a helpful resource for you.

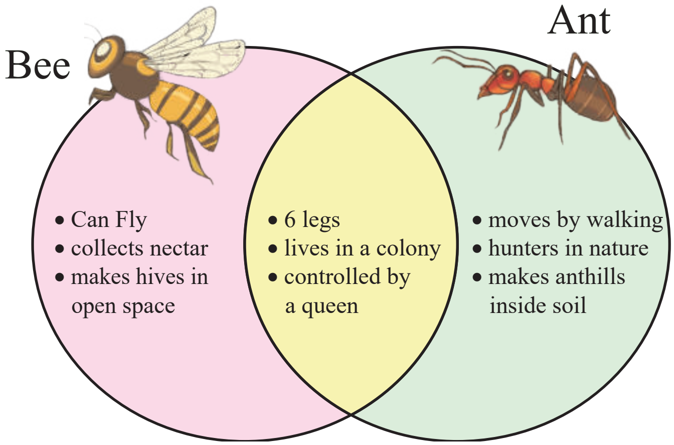
Good luck to you all.

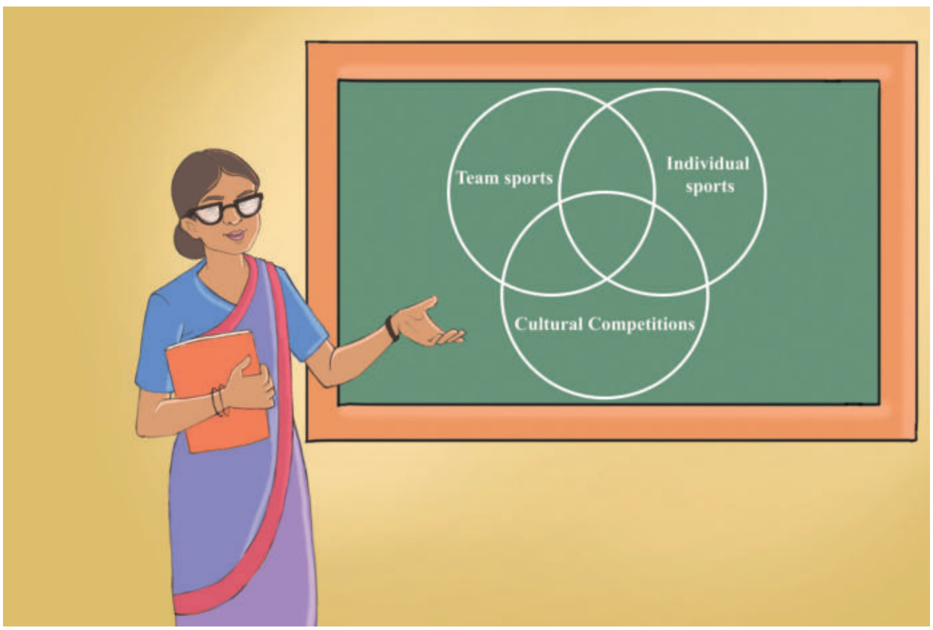
Index

Sets in Daily Life

You can learn from this experience-

- Concept of sets
- Types of sets
- Operations of sets
- Venn diagram
- Cartesian product
- Application of sets





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Sets in daily life

You are given a set of books on being promoted to a new class. When you started class VIII, what books were included in the set of books you were given? Write in the blank space below

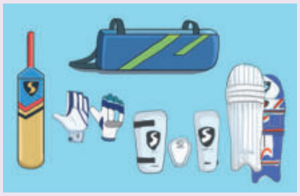
Set of books of class VIII:

Think about the last time you used coloured pencils, what colours were in your set of coloured pencils?

Colours in a set of coloured pencils:

Many of you must love to play or watch cricket. Below is a picture of a set of cricket playing equipments. Look at what are in the set and write in the blank next to it:

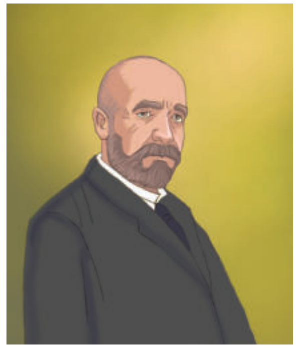
Set of cricket playing equipments:



By now you can understand that we are discussing sets of different things. You have seen the set of textbooks, color pencils, cricket equipment etc. A set can be of as many students as there are in your class. The colors of the national flag of Bangladesh can be a set. A set can also be made of whatever is on your reading table. These are real objects. Abstract objects can also be in sets. For example, the set of names of the players in your school's football team. There can also be different sets of numbers, for example, the set of integers.

Let's know

German mathematician Georg Cantor (Georg Ferdinand Ludwig Philipp Cantor) is known as the father of set theory. He was born in Russia. Cantor and his lifelong friend Richard Dedekind agreed after exchanging letters that a set is a collection of finite or infinite objects that have a certain property and each object retains its uniqueness.



Georg Cantor

Then we can say,

A specific collection of different objects, real or abstract, is called a set.

1.1 Importance of sets in mathematics

By now you may have started wondering what is the importance of sets in mathematics? A careful consideration of the following example will make this clear to you.

Example 1

Sixteen students of Mitu's school participated in a local Mathematical Olympiad, where students were given various quizzes to test their intelligence with a total score of 100. Based on the results, it was decided who among them would go to the National Mathematical Olympiad. Students who secured more than 60% marks would represent the school at the national level.

Name	Score	Name	Score	Name	Score	Name	Score
Sagor	58	Dipti	45	Koli	77	Maruf	50
Kamal	72	Avijit	63	Tribijoy	74	Andrew	76
Mitu	79	Jeba	90	Choity	81	Akash	59
Selim	33	Rowson	35	Nahar	78	Tasnim	80

Now, if we express the set of scores more than 60% by A and less than or equal to 60% by B, we get, A = {72, 79, 63, 90, 77, 74, 81, 78, 76, 80}

and

B = {58, 33, 45, 35, 50, 59}

What do we understand from this? Notice that we can understand the following matters clearly.

- More than half of the students got more than 60%.
- About one-third of the students got less than 60%.
- Scores less than 60% are between 33 and 59.

What other matters can be understood from these sets? Write in the blank space below

.

Note, in the above example we created two different sets based on a condition of some mathematical data. Now tell me what other decisions can be made to improve the skills of the students participating in Math Olympiad? Suppose we can draw the following conclusion.

The school and the mathematics teachers need to take urgent measures to improve the mathematical understanding of students whose scores are in the B set.

One such decision was made because we divided the students into two sets. Did you understand the need for sets through this example?

By set we can denote a collection of similar mathematical or abstract data. By separating similar information or data, it is possible to gain a clear understanding of data processing and relevant issues. So let us try to know about one such important matter.

1.2 Expressing the sets

By now you know the definition and importance of sets. There is also a nice way to express the set. The objects that will be expressed are separated by commas inside the second bracket. For example,

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The set of colours in the national flag of Bangladesh = {Green, Red}

Work in pairs

Express as a set:

1. The set of books of all subjects in class eight =
2. The set of colours in your colouring pencils =
3. The set of cricket equipments shown in the figure =

1.3 Method of writing sets

- Sets are usually named using capital letters A, B, C, ..., X, Y, Z of English alphabet.
- In a set, the objects are called elements. Elements are usually expressed using small letters $a, b, c, \ldots, x, y, z$ of English alphabet.
- If $B = \{a, b\}$, elements of set B are a and b. The symbol to express elements is $\in$, That is, $a \in B$ means that a is an element of set B or a belongs to B.
- If c is not an element of set B then we write $c \notin B$ and it is read as c is not an element of B or c does not belong to B.

Individual task:

1. Construct the set of prime factors of 210 and write in the empty box below.

Individual task

2. If $X = \{5, 7, 9, 11, 13\}$ write $\in$ or $\notin$ in appropriate boxes below.

$9 \ \boxed{\quad\quad} \ X$ $\qquad$ $10 \ \boxed{\quad\quad} \ X$

$3 \ \boxed{\quad\quad} \ X$ $\qquad$ $13 \ \boxed{\quad\quad} \ X$

1.4 Methods of expressing sets

As you can see, we can specifically express a collection of objects or numbers by means of sets. That is, it can be said precisely whether an object is an element of a set or not. For example-

- $A = \{1, 3, 5, 7, 9\}$ is the set of all odd positive numbers less than 10. Here we can specifically say which are the elements of A. For example- $3 \in A$ but $4 \notin A$.
- $B = \{a, e, i, o, u\}$ is the set of vowels in English alphabet. Here $i \in B$ but $b \notin$ B.

Sets are expressed in two ways. Roster Method or Tabular Method and Set Builder Method.

1.4.1 Roster Method or Tabular Method

In this method, all elements of the set are separated by commas and written in second brackets. For example,

- The set formed by numbers 1, 2, 3 : $A = \{1, 2, 3\}$
- The set of prime numbers: $P = \{2, 3, 5, 7, 11, \cdots\}$
- The set of even numbers: $E = \{\cdots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \cdots\}$

1.4.2 Set builder method

In this method all the elements of the set are expressed by specifice properties or conditions. For example,

$$A = \{x : x \text{ is an odd natural number}\}$$

Notice that, there is a ':' (colon) after x. The sign ':'is meant 'such that'. Since in this method rules are given for determining the elements of the set, this method is also called Rule Method.

Example 1. Express the set $A = \{0, 3, 6, 9, 12, 15\}$ in set builder method.

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Solution:

Here each element of the set is an integer, not less than 0, not greater than 15, and a multiple of 3. So in set builder method we can write,

$$A = \{x : x \text{ is an integer and multiple of } 3,\ 0 \le x \le 15\}$$

Example 2. Express the set $A = \{x : x \text{ is an integer, } x^2 \le 25\}$ in tabular method.

Solution: Here each element of the set is an integer whose square is less than or equal to 25. The numbers are 0, ±1, ±2, ±3, ±4, ±5. Therefore, in tabular method we can write,

$$A = \{0, \pm1, \pm2, \pm3, \pm4, \pm5\} = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

Individual task:

1. Express the following sets using set builder method.

   a) $A = \{-28, -21, -14, -7, 7, 14, 21, 28\}$

   b) $B = \{0, 1, 2, 3, 5, 8, \ldots\}$

2. Express the following sets using tabular method.

   a) $D = \{x{:}x,$ is a multiple of 5 and less than $30\}$

   b) $F = \{x{:}x$ is a factor of $30\}$

   c) $G = \{x{:}x$ is a positive integer and $x^2 < 17\}$

   d) $H = \{x : x^2 + 3x + 2 = 0\}$

Examples of some special sets

N : Set of all natural numbers

Z : Set of all integers

Q : Set of all rational numbers

R : Set of all real numbers

Z^+ : Set of all positive integers

Q^+ : Set of all positive rational numbers

R^+ : Set of all positive real numbers

1.5 Types of sets

1.5.1 Universal Set

If elements of any set are collected from a particular set, then the particular set from which the elements are collected is called the **universal set**. The universal set is usually denoted by U. But the universal set can also be expressed with other symbols. For example, If we consider the set of all even natural numbers $E = \{2, 4, 6, ...\}$ and the set of all natural numbers $N = \{1, 2, 3, 4, 5, 6, ...\}$ then N is the universal set of E.

Example: The set $A = \{x, y\}$ is from the set of English lowercase alphabet. So the set of English lowercase letters is the universal set of $A = \{x, y\}$.

1.5.2 Finite Set

A set containing a finite number of elements is called a finite set. For example-

1. $A = \{2, 4, 6, 8\}$
2. $B = \{a, e, i, o, u\}$
3. $F = \{x : x \text{ is a prime number and } 30 < x < 70\}$

The number of elements of set A and set B are respectively 4 and 5 .

Brain storm

What is the number of elements in set F ? Write how did you determine in the blank space below.

1.5.3 Infinite Set

A set whose numbers of elements is not finite is called an infinite set. We cannot finish counting the number of elements in an infinite set. For example,

a) $A = \{x : x \text{ is an odd natural number}\}$

b) Set of natural numbers $N = \{1, 2, 3, 4, ...\}$

c) Set of integers $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

d) Set of rational numbers $Q = \left\{\frac{a}{b} : a \text{ and } b \text{ are integers and } b \neq 0\right\}$

e) Set of real numbers R

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Brain storm

Why are the above sets infinite?

Team task

Some sets are described in the left-hand column of Table 1.1 below. Your task is to decide whether each set is finite or infinite by placing a tick (·) in the correct blank. Also write your logic in the blank column on the right side.

Table 1.1

Serial	Set	Finite	Infinite	Your logic
1	All odd numbers less than 10			
2	Rivers of Bangladesh			
3	Vowels of English alphabet			
4	Prime factors of 210			
5	$B = \{x : x \text{ is a factor of } 30\}$			
6	$D = \{x : x^2 + 3x + 2 = 0\}$			
7	$P = \{2, 3, 5, 7, 11, \cdots\}$			
8	$A = \{x \in N : 0 < x < 1\}$			
9	$B = \{x \in Q : x^2 = -1\}$			

Brain storm

While completing Table 1.1, were you able to determine the two sets at serial 8 and 9? Write the number of elements in these two sets in the space below.

Number of elements in set 8 =

Number of elements in set 9 =

1.5.4 Empty Set

The set which has no elements is called an empty set. It is denoted by Ø or { }. For example -

$A = \{x : 0 < x < 1$ where x is a natural number$\}$.

Again,

$B = \{x : x^2 = -1$, where x is a rational number$\}$.

Think if ɪt is possible or not. Write your answer in the space below.

Brain storm

- The set of boys in a girl's school is an empty set!
- The set of prime numbers which are also squares in an empty set.
- A = $\{x : x$ natural number, $8 < x^3 \leq 25\}$ is an empty set.

Now you and a classmate find five empty sets and write them .

1.5.5 Subset

If every element of a set A is an element of another set B, then set A is called a subset of set B and is written $A \subseteq B$ and read, A is a subset of B. Here $\subseteq$ denotes the subset.

Let $A = \{a, b\}$ be a set. The elements of this set form the sets $\{a, b\}$, $\{a\}$, $\{b\}$. Again, the empty set $\emptyset$ can be formed without taking any elements. Here, every element of the set consisting of $\{a, b\}$, $\{a\}$, $\{b\}$, $\emptyset$ is an element of the set A. So, each of the sets that can be formed from a set is called a subset of that set. The set A is equal to $\{a, b\}$ among the above subsets. Each set is its own subset. Again, the set $\emptyset$ can be formed from any set. So $\emptyset$ is a subset of any set.

Think: Can a universal set be its own subset?

Verify

1. Suppose, $P = \{1, 2, 3\}$, $Q = \{2, 3\}$, and $R = \{1, 3\}$

   a) Q and R are subsets of P, because ______________________________

   b) Q is a subset of P. It is expressed as: ____________________________

   c) $P \subseteq P$, is this expression true or false? Give logic behind your answer: ______

   ____

2 $N \subseteq N$, where N is the set of natural numbers.

1.5.6 Equal set

Suppose, A and B are two sets, where

$A = \{6, 7, 8, 9\}$ and $B = \{6, 9, 8, 7\}$

Notice the elements of A and B. Are they the same?

Can we claim that $A = B$?

Brain storm

Think and answer with logic if the following claims are true or not.

1. $A \subseteq B$

2. $B \subseteq A$

If two sets have same elements, then they are called equal sets. If A and B are two sets where, $A \subseteq B$ and $B \subseteq A$, then A and B are equal sets and we write $A = B$.

EXAMPLE: $A = \{3, 5, 7\}$ and $B = \{5, 3, 7\}$ are two equal sets. Here, $A = B$ because $A \subseteq B$ and $B \subseteq A$.

Again, if $A = \{3, 5, 7\}$, $B = \{5, 3, 3, 7\}$ and $C = \{7, 7, 3, 5, 5\}$, the sets A, B and C are equal. That is, $A = B = C$

👉 Changing the order of the elements of the set or repeating an element does not change the set.

Check yourself!

Elements of sets A and B are given below. Check which pairs contain equal sets.

1. $A = \{6, 7, 8, 9\}$ and $B = \{6, 9, 8, 7\}$
2. $A = \{4, 8, 6, 2\}$ and B $=\{x : x$ is a positive even number and $x < 10\}$
3. $A = \{-1, -2\}$ and $B = \{x : x^2 + 3x + 2 = 0\}$

1.5.7 Proper subset

Each set is a subset of itself. Let A be a set. Any subset of A other than A is called a proper subset of A. The proper subset is denoted by the symbol $\subset$. So if B is a proper subset of A then we write $B \subset A$. That is, $B \subseteq A$ but $B \neq A$. The number of elements of a proper subset formed from a finite set will be less than the number of elements of the given set.

Example: Suppose, $A = \{ 3, 4, 5, 6\}$ and $B = \{3, 5\}$ are two sets. Here $B \subseteq A$ but $B \neq$ A. Therefore B is a proper subset of A.

Problem: Write all the subsets of $P = \{x, y, z\}$ and find out the proper subsets.

Solution: Given that, $P = \{x, y, z\}$

Subsets of P: $\{x, y, z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x\}$, $\{y\}$, $\{z\}$, $\varnothing$

Proper subsets of P: $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x\}$, $\{y\}$, $\{z\}$, $\varnothing$

Brain storm

1. Which set is a subset of all sets?
2. Which set has maximum only one subset?
3. Suppose, $A = \{1, 2, 3\}$. How many subsets are there of A and what are they?
4. Verify $2Z \subset Z$, where Z is the set of all integers

If there are n number of elements in a set then the numbers of subsets of that set is 2^n and the numbers of proper subsets of that set is $2^n - 1$.

Verify and write in your notebook

Suppose, $P = \{1, 2, 3\}$, $Q = \{2, 3\}$, and $R = \{1, 3\}$

1. Are Q and R proper subsets of P? Give logic behind your answer.
2. If Q is a proper subset of P, we express this as:
3. How many proper subsets of P are there? Find them

1.5.8 Set of sets!

Suppose there are 18 boys and 22 girls in your class and 23 boys and 19 girls in class eight.

Let A be the set of boys and B be the set of girls in class 9 where set C be the set of boys and D be the set of girls in class 8. Then in set builder method we can write,

Set of boys in class 9, $A = \{x : x \text{ is a boy in class } 9\}$

Then, what would be the set of class 9 girls and sets of class 8 boys and girls expressed in set builder method? Write in the space below.

Now if we form a set X with the set of boys and girls of class 9 and 8, then we can write,

$X = \{A, B, C, D\}$

Here X is a set of sets. In this case the set A, is an element of X. That is, $A \in X$.

Example:

$X = \{\{0, 1\},\{1, 2, 3\},\{0, 1, 3\}\}$ is a set of sets. Here $\{0, 1\} \in X$ but $0 \notin X$.

1.5.9 Power Set

Consider a set $A = \{x, y\}$. Then the subsets of set A are $\{x, y\}$, $\{x\}$, $\{y\}$ and $\emptyset$. Here the set of subsets is $\{\{x, y\}, \{x\}, \{y\}, \emptyset\}$, the power set of set A. So, the set formed by all the subsets of a set is called the power set of that set. The power set of A is denoted by $P(A)$.

Example: If $A = \{0, 1, 2\}$ determine $P(A)$.

Solution:

Here the subsets of the set A = $\{0, 1, 2\}$ are: $\emptyset$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$, $\{0, 1, 2\}$.

Therefore, $P(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Work in pairs: Find the power sets of the following sets. One is done for you.

1. $A = \emptyset$. Here A has only one subset $\emptyset$. Therefore, $P(A) = \{\emptyset\}$.
2. $B = \{a\}$.
3. A pen holder contains a pen, a pencil and an eraser. C is a set containing them .

1.6 Number of elements of a set

The number of elements in a set plays an important role in the use of sets. The number of elements of any set A is denoted by $n(A)$. If A is an infinite set, then $n(A)$ is denoted by ∞. That is, if A is an infinite set, then $n(A) = \infty$.

Example: If $A = \{0,1,2,3\}$, $n(A) = 4$.

Individual task

1. If $A = \{0\}$, $n(A) = \boxed{}$
2. If $A = \{a, b, c\}$, $n(\text{A}) = \boxed{}$
3. If $A = \emptyset$, $n(A) = \boxed{}$
4. If N is the set of all natural numbers, $n(N) = \boxed{}$

Now let's look at the number of elements in the power set.

Note: If the number of elements of a set is n, the number of subsets of that set will be 2^n. So, the number of element s in the power set will be 2^n.

Individual task

If $A = \{0, 1, 2, 3\}$, determine $P(A)$. Show that number of elements in, $P(A)$ is $2^4 = 16$.

1.7 Operation of sets

In the case of numbers, there is addition, subtraction, multiplication, division. These are called arithmetic operations. Similarly, there are operations for sets. One or more sets can be constructed from another set. Now we will discuss set operations.

1.7.1 Union of Sets

As you noted earlier, elements of two sets can have similarities and differences. Some operations can be done between the sets to make decisions based on these similarities and differences. An example of a situation will help you understand. In ninth grade 4 students like to play football and 3 students like to play basketball. Suppose, the set of roll numbers for those who like football $A = \{3, 4, 5, 6\}$ and

the set of roll numbers for those who like basketball is $B = \{1, 4, 6\}$.

Now, what is the set of roll numbers for those who like to play either football or basketball and how do we express this set? This set is denoted by $A \cup B$ and all elements of A and B form $A \cup B$.

That is,

$$A \cup B = \{1, 3, 4, 5, 6\}$$

A set consisting of all the elements of two or more sets is called the union of the sets. Suppose A and B are two sets. The union set of sets A and B is denoted by $A \cup B$ and is read as A union B. The set construction method is written as,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Example: $A = \{x : x \in Z, -2 < x < 5\}$ and $B = \{1, 4, 6, 8\}$. Determine $A \cup B$.

Solution: According to question $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \{1, 4, 6, 8\}$.

Therefore, $A \cup B = \{-1, 0, 1, 2, 3, 4\} \cup \{1, 4, 6, 8\} = \{-1, 0, 1, 2, 3, 4, 6, 8\}$

1.7.2 Intersection of Sets

Now say what is the set of roll numbers of students who like both football and basketball in class 9 like in the above example. Also, how do we express this set?

This set is denoted by $A \cap B$ and it consists of the common elements of A and B. That is, the set of roll numbers of students who like both football and basketball in class IX is

$$A \cap B = \{4, 6\}$$

A set consisting of elements common to two or more sets is called an intersection set. Suppose A and B are two sets. The intersection of sets A and B is denoted by $A \cap B$ and is read as A intersects B or A intersects B. In set builder method it is written as,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

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Example: $X = \{x \in Z : -4 < x < 8\}$ and $Y = \{x \in N: x$ is even, $x \leq 18\}$. Determine $X \cap Y$.

Solution:

Given that, $X = \{x \in Z: -4 < x < 8\} = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

and $Y = \{x \in N : x$ is even, $x \leq 18\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

Therefore, $X \cap Y = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\} \cap \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ $= \{2, 4, 6\}$

1.7.3 Difference of two sets

Mathematics teacher selected 5 students from Sonapur Madrasah for Mathematical Olympiad from students of class 9. They are Samir, Nasrin, Tahsin, Bashir and Amina. On the other hand, the Arabic teacher selected 3 students to participate in the Quran recitation competition. They are – Kohinoor, Bashir and Rezwan. If the Math Olympiad set is A and the Quran recitation set is B, then we can write,

A = {Samir,Nasreen,Tahsin,Bashir,Amina} and B={Kohinur,Bashir,Rezwan}

As the two competitions were held on the same day, the headmaster said that the members of set B should be eliminated from set A. So how do we express the set after excluding B from A and who are the members of the set?

This set is denoted by $A \setminus B$ and $A \setminus B$ is formed by removing the members of B from A. That is,

$A \setminus B$={Samir,Nasrin,Tahsin,Amina}

Here Bashir is removed from the difference set, as he is also a member of set B.

> A set formed by excluding members of one set from another set is called a set difference. The difference of set A and set B is denoted by $A \setminus B$ and is read as A difference B. In set builder method it is written as,
>
> $$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

Example: $P = \{x : x$, 12 is a factor of 12$\}$ and $Q = \{x : x$ is a multiple of 3, $x \leq 12\}$. Determine $P \setminus Q$.

Solution: Here, $P = \{x: x$, is a factor of 12$\} = \{1, 2, 3, 4, 6, 12\}$

and $Q = \{x : x$, is a multiple of 3, $\leq 12\} = \{3, 6, 9, 12\}$.

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Therefore $P \setminus Q = \{1, 2, 3, 4, 6, 12\} \setminus \{3, 6, 9, 12\} = \{1, 2, 4\}$

Brain storm

1. If $A = \{0, 1, 2, 3, 4\}$ and $B = \{1, 2, 2, 3, 1\}$, then $B \setminus A$ =? Explain.

2. If $A \subseteq B$, then $A \setminus B$ = ? Explain.

1.7.4 Complement of a Set

Let U be the set of population of the entire world and A be the set of population of those who speak in Bangla. Then U is the universal set and A is a subset of U. Now, how can we express population of the world without any who speaks in Bangla?

This set is denoted by $U \setminus A$ and $U \setminus A$ is formed by removing the members of A from U. That is, $U \setminus A$ is the set of people who do not speak in Bangla

The set formed by excluding the elements of a set A from the elements of its universal set U is called the complement set of A. The complement of a set A is denoted by A^c or A' and is read as the complement of A. In set builder method it is written as,

$$A^c = \{x : x \in U \text{ and } x \notin A\}$$

Example: If universal set U is the set of all digits, and A is the set of all even digits, determine A^c.

Solution:

Here, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{0, 2, 4, 6, 8\}$

So, $A^c = \{1, 3, 5, 7, 9\}$

1.7.5 Disjoint Set

There is a small group of students in a school and they are nine memers in the group. Roll numbers of group members are very interesting, which are the first 9 prime numbers. Some of the members sing, some dance. Those who neither dance nor sing, encourage others. They want to present a group presentation in a school co-curricular activity.

See below who does what, by their roll numbers.

If U is the set of roll numbers of the group members, write U in the box by tabular method:

$U =$

Set of roll number of students who sing, $E = \{5, 11, 17, 23\}$

Set of roll number of students who dance, $F = \{2, 7, 13\}$

Notice that, $F = \{2, 7, 13\}$, that is, it is not possible for them to give a common group presentation. In such a case it can be said that E and F are disjoint sets.

Two sets A and B are called disjoint sets if $A \cap B = \emptyset$

Check yourself!

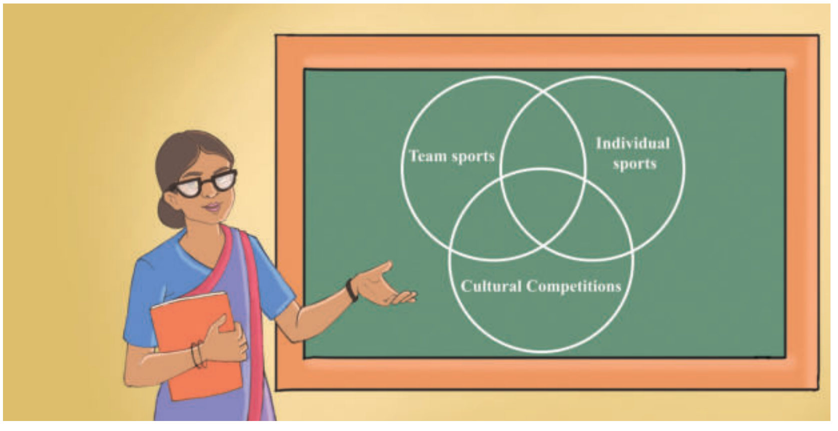
Given the above problem, determine the two sets below and write what do the sets indicate in terms of the group.

1. $E^c \cup F^c$

2. $E^c \cap F^c$

1.8 Solving a sports problem using diagram

Annual sports and cultural competitions will be organized in Nitu's school. The class teacher will take names of the participants in various events. The condition is that no student from class nine can participate in more than three activities. Teacher said, "Not everyone will participate in all three events. We must come to a decision. Let's say we have team sports, individual sports, and cultural activities." By saying this, she drew three circles on the board like the picture below.



Then she said, “Who wants to participate only in team sports, like cricket or football?” Eight people in the class who are in different sports teams of the school raised their hands and teacher wrote their roll numbers in the team sports circle. In the same way, she wrote the roll numbers of the students who wanted to participate in individual sports and cultural competitions in the correct circle.

Then she asked, “Is there anyone who wants to participate in both team and individual sports?” Utsho, Sharif and Nazmul were in the football team, they want to participate in the race. Again, Seema and Aparna named in individual sports, they also want to play volleyball. The roll of these five students were erased from team and individual and written in the place where the circles of group and individual intersected each other.

In this way, the roll numbers of five and six people in team sports and cultural activities, and individual sports and cultural activities were placed. At the end she asked, “Now tell me is there someone who wants to participate in all three events?” Utsho quickly raised his hand and said, “I want to recite a poem.” Apa said, “Very good Utsho!” Then she wrote the roll number of Utsho in the box where the three circles intersect.

You can notice how easily the teacher found a simple solution to a complex problem! Not only that, but a visual representation was also seen on the board. Representation in this way is called Venn diagram.

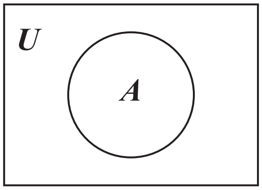
1.8.1 Venn Diagram


John Venn

The Venn diagram is named after its inventor, the English philosopher and logician John Venn.

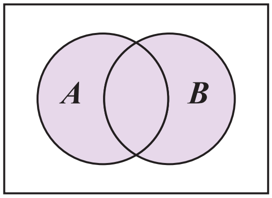
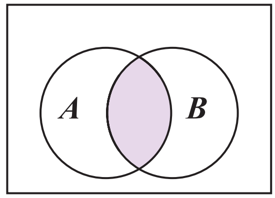
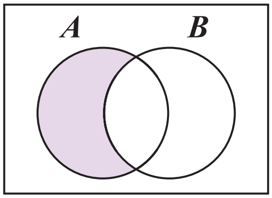
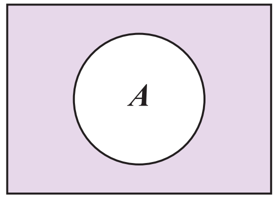
In a Venn diagram, the universal set is represented by a rectangular geometric shape on a plane, and the subsets of that universal set are represented by circles inside that rectangular area. The adjacent Venn diagram shows the universal set U and a subset A of it.

How set operations can be represented using a Venn diagram is shown below. Later, we will use Venn diagram to express the elements of the set.



1.8.2 Set operations using Venn diagram

For any sets A and B, the Venn diagrams of $A \cup B$, $A \cap B$, $A \setminus B$ and A^c are shown below.

 $A \cup B$	 $A \cap B$
 $A \backslash B$	 A^c

1.8.3 Venn diagram in real life problems

Problem-1. Hilsa fish is in the list of favourities

It is difficult to find a Bengali person who does not like Hilsa fish. Surely there are people who are not Bangladeshi but like Hilsa fish. Can you imagine the set of people who are not Bangladeshi but like Hilsa fish? Let's do a little analysis

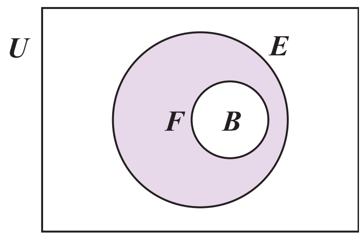
Suppose, set of world population is U.

The set of Hilsa fish liking population in the world is E.

The set of people who are not Bangladeshi but like Hilsa fish is F.

The set of Bangladeshi who like Hilsa fish is B.

The set of people who are not Bangladeshi but like Hilsa fish is represented by Venn diagram.



$F = E \backslash B$

Brain storm

For any sets A, B, C express the following sets using Venn diagram.

1. $A \cup B \cup C$
2. $(A \cap B)^c$
3. $A \cap (B \cup C)$

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Problem-2. Characteristics of ants and bees

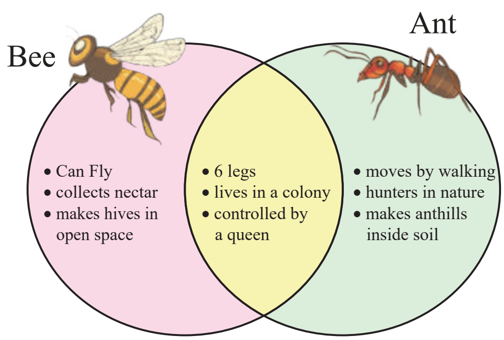
Below are some characteristics of ants and bees. Find out their common characteristics through Venn diagram.

 Some characteristics of bees	 Some characteristics of ants
• 6 legs • lives in a colony • can fly • collects nectar • controlled by a queen • makes hives in open space	• 6 legs • moves by walking • lives in a colony • hunters in nature • makes anthills inside soil • controlled by a queen

Solution: Suppose, A is the set of characteristics of bees and B is the set of characteristics of ants.

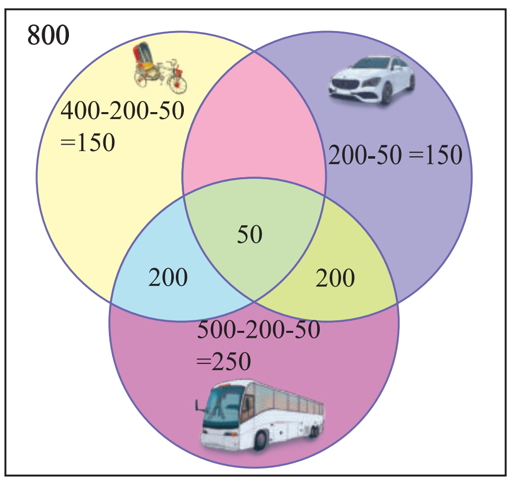
Their characteristics are represented by a Venn diagram such that their common characteristics lie in A∩B. Common characteristics as per the Venn diagram on the side are:

A ∩ B={6 legs, lives in a colony, controlled by a queen }.



Problem-3. Transportation system

A survey of 800 people in a city shows that 500 people travel by bus, 200 people travel by car, 400 people travel by rickshaw, 200 people travel by both bus and rickshaw but not by car and 50 People travel by buses, rickshaws, and cars. Others commute on foot. Find out how many people commute on foot by representing it in a Venn diagram.



Solution: Suppose, U is the set of people under survey, B is the set of people travelling by bus, C is the set of people travelling by cars, R is the set of people travelling by rickshaws and W is the set of people commuting on foot.Then, $n(U) = 800$, $n(B) = 500$, $n(C) = 200$, $n(R) = 400$.

According to the Venn diagram, set of people using all three vehicles is $B \cap R \cap C$

$$\therefore (B \cap R \cap C) = 50$$

Number of people who travel both by bus and rickshaw but not by car

$$n(B \cap R \cap C^c) = 200$$

People who travel only by bus

$n(B) - n(B \cap R \cap C^c) - n(B \cap R \cap C)$

$= (500-200-50)$

$= (500-250) = 250$

People who travel only by rickshaw $n(R) - n(B \cap R \cap C^c) - n(B \cap R \cap C)$

Calculate in the following box

People who travel only by cars $n(C) - n(B \cap R \cap C)$

$= (200 - 50) = 150$

$\therefore$ People who travel using at least one transport
$n(B \cup R \cup C)$ [Calculate in the following box]

So, people who commute only by walking $n(W) = n(U) - n(B \cup R \cup C)$

$= (800 - 800)$

$= 0$

So, nobody commutes on foot.

1.9 Cartesian product of sets

Let A be a set of colors with two types, namely white and black, that is A = {white,black} and B be a set of clothes with three types of clothes, namely shirt, pants, panjabi, that is B = {shirt ,Pants, Panjabi}. So, we can create a new set in the order of taking colors first and clothes later. This set is denoted by $A \times B$. Now the question is, how many elements can we make in this order: first color and then clothing? Note the table below. Here the elements of $A \times B$ are created as ordered pairs using color on one side and clothing set on the other, like two-dimensional Cartesian coordinates.

	Shirt	**Pant**	**Panjabi**
White	(White,Shirt)	(White,Pant)	(White,Panjabi)
Black	(Black,Shirt)	(Black,Pant)	(Black,Panjabi)

that is,

$A \times B$ = {(White, Shirt), (White, Pant), (White, Panjabi), (Black, Shirt), (Black, Pant), (Black, Panjabi)}

So,we can write

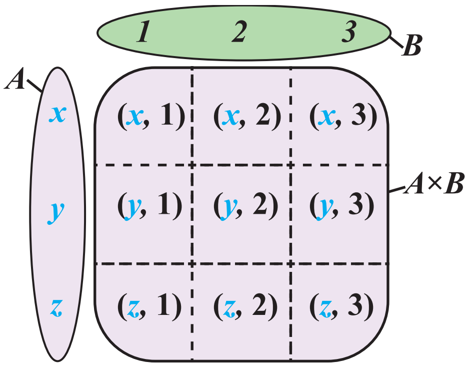
$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Example: If A = $\{x, y, z\}$ and B = {1, 2, 3}, then

A × B = {(x, 1), (x, 2), (x,3), (y,1), (y,2), (y,3), (z,1), (z,2), (D,3)}

Below we give table 1.2,

Table 1.2



Notice that: (i) $(x, 1) \in A \times B$ but $(1, x) \notin A \times B$.

(ii) $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$.

(iii) For any set A , $A \times \emptyset = \emptyset$.

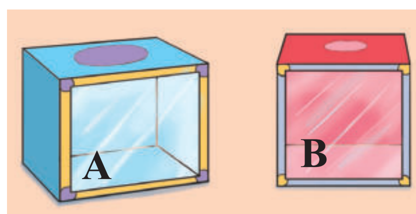
Brain storm

1. $n(A) = 3$ and $n(B) = 2$, $n(A \times B) = ?$

2. If $n(A) = p$ and $n(A \times B) = ?$

1.10 Team task/project

The teacher will write the names of different games on small pieces of paper and put them in two boxes (A and B). Each box will contain at least 5 names of games.



Now the students will be divided into groups of 9 (or any odd number) as per the instruction of the teacher. The team leader of each team will pick a total of two games from the boxes A and B and ask the other team members which of the two lottery games they like to play.

Their possible answers are: (a) Both (A and B), (b) One of them (A or B), (c) Neither. Now make a list of names in the notebook of who likes A, who likes B, and who likes neither. If someone likes both sports, his/her name will be in both list A and B. Now complete the following tasks and present them to the class.

1. Present the following sets from your collected data. in tabular method

    a) U = {Names of all the students in the group from whom information is taken}

    b) A = {Those who like games from group A}

    c) B = {Those who like games from group B}

2. Present the above data from 'a', 'b' and 'c' in a Venn diagram. Mention those who prefer neither A nor B also in the Venn diagram.

3. Now fill the table below with the numbers obtained from the set operations.

$n(A)$	
$n(B)$	
$n(A \cap B)$	
$n(A \cup B)$	
$n(\cup)$	

$n(A^c)$	
$n(B^c)$	
$n(A \cup B)^c$	
$n(A \cap B)^c$	

4. Verify the following identities:

    a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

    b) $n(U) = n(A) + n(A^c)$

    c) $n(A \setminus B) = n(A) - n(A \cup B)$

    d) $n(A^c \cap B^c) = n(U) - n(A \cup B)$

Conclusion

Many important discoveries in the applied branch of mathematics are based on Georg Cantor's set theory, which you will learn in higher secondary classes and universities. In this class you learnt how to read sets, methods of expression, different types of sets, different types of subsets, expression using Venn diagrams and use of ordered pairs. Hopefully, these exercises will expand your world of thinking and analysis and allow you to apply these skills in real life to solve complex problems.

Exercises

1. Express in tabular method:

   a) $A = \{x \in N: -3 < x \leq 5\}$

   b) $B = \{x \in Z: x \text{ is a prime number and } x^2 \leq 50\}$

   c) $C = \{x \in Z: x^4 < 264\}$

2. Express in set builder method:

   a) $A = \{1, 3, 5, \ldots, 101\}$

   b) $B = \{4, 9, 16, 25, 36, 49, 64, 81, 100\}$

3. If $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 1, 3, 5, 6\}$ and $C = \{1, 5, 6\}$, then find the following sets.

   a) $A \cup B$ b) $A \cap C$c) $B \backslash C$ d) $A \cup (B \cap C)$ e) $A \cap (B \cup C)$

4. If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7\}$, $B =\{0, 2, 4, 6\}$ and $C = \{3, 4, 5, 6, 7]$, then verify the following relations:

   a) $(A \cup B)^c = A^c \cap B^c$ b) $(B \cap C)^c = B^c \cup C^c$

   c) $(A \cup B) \cap C = (A \cap C) \cap (B \cap C)$ d) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

5. Find the following:

   a) $N \cap 2N$ b) $N \cap A$ c) $2N \cap P$

   Where, N is the set of all natural numbers, A is the set of all odd numbers, P is the set of all prime numbers.

6. Let U be the set of all triangles and A be the set of all right triangles. Then describe the set A^c.

7. Show the followings with Venn diagrams. For any sets A, B, C -

   a) $(A \cup B)^c = A^c \cap B^c$ b) $(B \cap C)^c = B^c \cup C^c$

   c) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ d) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

8. Out of 40 students in a class, 25 like birds and 15 like cats. There are 10 students who like both birds and cats. Determine by a Venn diagram how many students like neither bird nor cat.

9. If $P = \{a, b\}$, $Q = \{0, 1, 2\}$ and $R = \{0, 1, a\}$, then find the values of expressions below.

   a) $P \times Q$, $P \times P$, $Q \times Q$, $Q \times P$ and $P \times \emptyset$

   b) $(P \times Q) \cap (P \times R)$

   c) $P \times (Q \cap R)$

   d) $(P \times Q) \cap R$

   e) $n(P \times Q)$, $n(Q \times Q)$

   Give your logic on the equality of (c) and (d).

10. If $P = \{0, 1, 2, 3\}$, $Q = \{1, 3, 4\}$ and $R = P \cap Q$,

    (i) Determine $P \times R$ and $R \times Q$.

    (ii) Find the values of $n(P \times R)$ and $n(R \times Q)$.

11. If $P \times Q = \{(0, a), (1, c), (2, b)\}$ then determine P and Q.

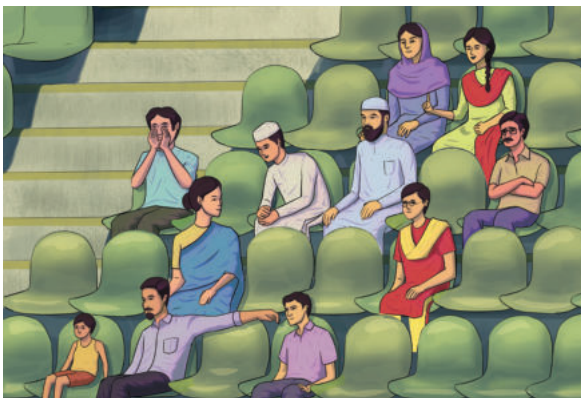
Sequence and series

You can learn from this experience-

- sequence
- Arithmetic sequence
- Geometric sequence
- Fibonacci sequence
- series
- Arithmetic series
- Geometric series

Sequence and series

The word “sequence” is indeed quite a familiar word in your daily life, isn’t it? You have to organize many things in your daily routine in a sequence. Think about your study table or the bookshelf next to you. Typically, the largest books are placed at the bottom, followed by smaller ones stacked on top. Then, in a sequence, you arrange the smaller books upward. Before the beginning of your school classes, there is usually a gathering where everyone follows a specific order. You must stand according to your height The class teacher calls you by roll numbers after the assembly ends. How are your roll numbers arranged, in a sequence, right? Besides your school, you must have noticed that in the market, some shopkeepers arrange their products in various ways. For example, the fruit vendor arranges apples and oranges in a pyramid-like fashion. Similarly, the potter, the utensil seller, the ball seller, and the toy seller arrange their items from larger to smaller in a sequential manner. Think about the seating arrangement in a sports stadium. Even in a cinema hall, how are the seats arranged? Take a close look at these two pictures below.



Are there any distinctive features in the arrangement of chairs and pots? Discuss with your classmates. If you have found some specific features in the seating arrangement and pots made of clay, please write them in the empty box below.

We observe various types of sequences in different sectors around us and it is from these sequences that the concept of order primarily arises. Besides, you have already learned a lot about number patterns. For example, you can think about natural numbers 1, 2, 3, 4, ... … … These numbers can have a few more distinct characteristics apart from being arranged in order. Think about what other characteristics these numbers might have. Quickly jot down these characteristics in the empty box below:

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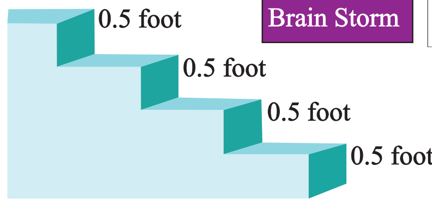


Brain Storm

What is height of each step from the surface of the staircase shown in the figure? Are their any spacific characteristics among the obtained answers?

Let us play two interesting games:

1. Earning Pocket Money

Imagine your family wants to give you some pocket money for a month. They have given you three options for how you can receive your pocket money, and you have to choose one of them. Here are the options:

A) Receive 10 Taka every day.

B) Receive 3 Taka on the first day, 3.50 Taka on the second day, 4 Taka on the third day, and so on, increasing by 50 paisa every day.

C) Receive 1 Taka on the first day, 2 Taka on the second day, 4 Taka on the third day, and so on, doubling the amount every day.

You need to choose one of these options and present your choice with arguments and explanations.

2.Prime Number Game

You need to find at least three prime numbers and there are some conditions:The difference between any two numbers must be either the same or equal and you need to fill in the empty cells satisfying the conditions. Explain the reason if you do not get three numbers satisfying the conditions.

General difference	1^{st} no.	2nd no.	3rd no.	...
2	3	5	7	
4				
9				
10				
14				
20				

Till now, you've been thinking about topics where there are special characteristics involved in arranging or obtaining numbers or objects, haven't you? These characteristics or rules are what we call patterns. You've learned about patterns in previous classes.

In a fun game of getting your hand spent –

a) In this month every day the hand costs you have got is the following

Day	1	2	3	4	…	30
Taka	10	10	10	10	…	10

b) In this month every day the hand costs you have got is the followin

Day	1	2	3	4	…	30
Taka	3.00	3.50	4.00	4.50	…	18.50

c) In this month every day the hand costs you have got is the following

Day	1	2	3	4	…	30
Taka	1	2	4	8	…	536,870,912

If you look at the above examples, you will see that in each case the hand cost receipt is related to the number of days {1,2,3,4…30}. The number of days {1,2,3,4…30} is a finite set of natural numbers. Here there is a relationship between the number of days and the amount of money. This relationship is a sequence.

The relation of a set of natural numbers to another collection is called a sequence.

In the above example the daily the collection of daily amount of money is a sequence. Each element of a sequence is called a term. The first element of a sequence is called the first term, the second element is called the second term, the third element is called the third term, and so on, in sequential order. If n is a natural number, we call the common term in a sequence the nth term. If you have a sequence with the first term a_1, the second term a_2, the third term a_3, and so on, up to the nth term a_n, we can write the sequence as $a_1, a_2, a_3, ..., a_n, \dots$. It is termed as (a_n)

Example 01: 1, 1, 1, 1, ,1,… is a sequence whose nth term $a_n = 1$. This is a constant sequence. Can you tell me the reason? Notice that every term in this sequence is the same. There is no change. This type of sequence is called constant sequence.

Individual work: Give two examples of constant sequence and write the nth term of each.

Individual work: Give two examples of periodic sequence and write the nth term of each.

Example 02 : 1, -1, 1, -1,… is a sequence whose nth term is $a_n = (-1)^{n+1}$. It is denoted by $((-1)^{n+1})$. It is a periodic sequence. Can you tell me the reason? Take a look, how the terms of this sequence are coming? Here two terms are recurring. That is why this type of sequence is called periodic sequence. The tides that occur daily in real life are an example of a periodic sequence.

Example 03 : 1, 3, 6, 10 ,… is a sequence. This is called a sequence of equilateral triangle numbers, because the terms of the sequence come from the numbers of the equilateral triangle-shaped balls in the adjacent figure.

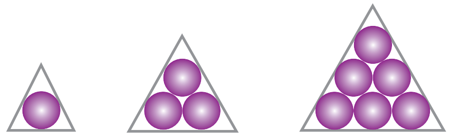


Fig.-1(a) Fig.-1(b) Fig.-1(c)

Individual work: Write the sequence of square numbers and represent them using squares

Example: 4. 1, 3, 5, 7, ..., $(2n - 1)$, ... is a sequence whose nth term is $a_n = 2n - 1$. This is a sequence of odd numbers.

Example 5: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$is a sequence whose nth term is $a_n = \frac{n}{n+1}$.

It is denoted by $\left(\frac{n}{n+1}\right)$

Example 6: $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots, \frac{n}{2n-1} \ldots$ is a sequence whose nth term is $a_n = \frac{n}{2n-1}$. Can you tell by what it would be indicated?

Group Task:

A) Determine the common term of the following sequences:

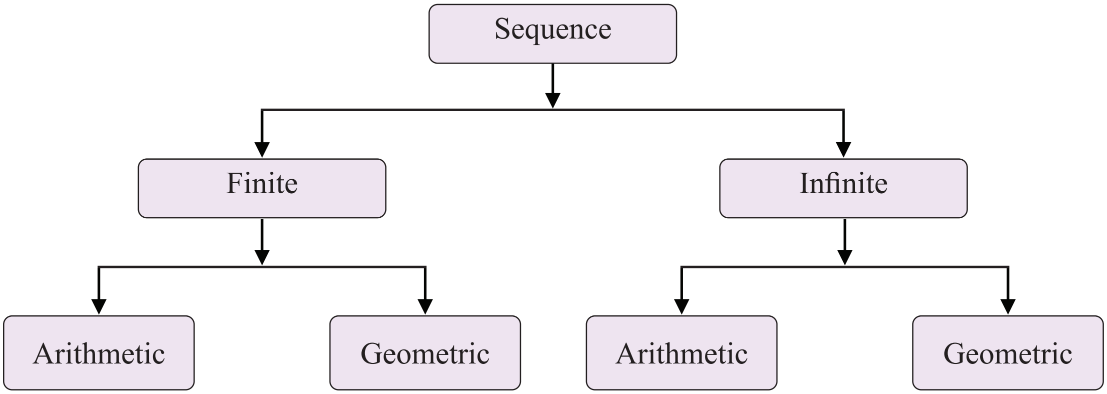
i) 3, 6, 9,... ii) 5, −25, 125, −625, ...

iii) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ iv) $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{4}{2^4} \ldots$

B) Find the sequences from the given common terms:

i) $\frac{n-1}{n+1}$ ii) $(-1)^{n+1} \frac{n}{n+1}$ iii) $(-1)^{n-1} \frac{n}{2n+1}$ iv) $\frac{n^2}{2n^2-1}$

Classification of sequence:



It needs to be mentioned, the number of terms in a sequence can be finite or infinite. A sequence with a specified number of terms, meaning it has an ending, is called a **finite sequence**. On the other hand, a sequence with an unspecified number of terms, meaning it has no ending, is called an **infinite sequence**.

Example:

Finite Sequence	Infinite Sequence
i) $1, 4, 9, \ldots, 100$	i) $3, 1, -1, -3, \ldots$
ii) $7, 12, 17, \ldots, 502$	ii) $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$
iii) $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots, \frac{1}{10001}$	iii) Multiple of 5 $= 5, 10, 15, \ldots$
iv)) Sunday, Monday, Tuesday, ... Saturday	iv) Counting Numbers $= 1, 2, 3, \ldots$

Brain storm

Determine the next terms in the sequence

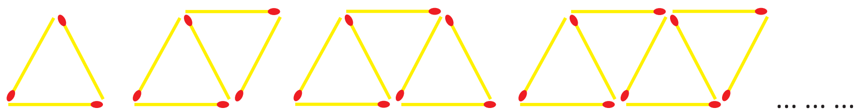
i) $-1, 2, 5, 8,$ ______ , ______, ______. ii) 3.4, 4.5, 5.6, ______ , ______, ______.

Arithmetic Sequence

Let's start by analyzing a few events or examples:

Event 1

Attach the matchstick to the exercise book and create a pattern like the one below



Now try to find the answers to the following questions:

i) How many sticks are needed for each of the pictures? Write the numbers in the adjacent boxes in order:	______ , ______, ______, ...
ii) Determine the difference between two consecutive numbers and write it in the adjacent box.	
iii) Have you found any common characteristics among the numbers? Write your comments in the box.	

Event - 2

Suppose, you have completed your education and joined a job. Your initial monthly salary is 25,000 Tk and you receive an annual increment of 500 Tk. So, in the 1^{st}, 2^{nd}, and 3rd years, your monthly salary will be 25,000 Tk, 25,500 Tk, and 26,000 Tk, respectively. Now, if you calculate the difference in your monthly salaries for consecutive two years, you'll notice that your salary increases by 500 Tk every year.

By the way, what similarities have you observed between these two scenarios or examples? Write down the similarities in the empty box below:

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Individual Task:

Notice the following sequences. Write about the characteristics of terms of each sequence.		
Sl. No.	Sequence	Characteristics
i)	4, 7, 10, 13,…………	
ii)	−2, −6, −10, −14,………	
iii)	$\frac{1}{2}$, 1, $\frac{3}{2}$, 2,………	

After doing the task, you must have known that there is a common difference between every two consecutive terms in each sequence, right? One of the terms is created by adding or subtracting a specific number to/from the preceding term, and the same property exists when generating any term after a given term in the sequence. Sequences that follow this kind of property are called **Arithmetic Sequences**. The first term of an arithmetic sequence is denoted as a_1, and the common difference is denoted as d. Therefore, we can express an arithmetic sequence in the following mathematical form:

Algebraic form of arithmetic sequence

$$a, a + d, a + 2d, a + 3d, \ldots$$

The first term of the sequence a and common difference d because --

2^{nd} term – 1^{st} term $= a + d - a = d$

3^{rd} term – 2^{nd} term $= a + 2d - (a + d) = a + 2d - a - d = d$

4^{th} term – 3^{rd} term$= a + 3d - (a + 2d) = a + 3d - a - 2d = d$

In a similar way, sutracting previous term from any term you will get d

Now tell me what will be the general term of the sequence? Observe that the general term will be,

$$a_n = a + (n - 1)d$$

Group Work:

Discuss in groups all the sequences below in terms of “which one is arithmetic and which one is not” including calculations. In your notebook, create a table like the one below and put a checkmark (√) next to the arithmetic sequence and a cross mark (×) next to the non-arithmetic sequence. Then, present in the class.

Sequence	Arithmetic	Non Arithmetic	If Arithmetic, What is common difference	Logical explanation
i) −4, 3, 10, 17,...				
ii) 1, 4, 9, 16,...				
iii) $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots$				
iv) $x - 3, x - 5, x - 7, \ldots$				

Determine general term or nth term of Arithmetic sequence

Find the common difference and the next three terms of the following sequences. A hint is provided for you:

Sequence	common difference	Sequence including the next three terms
i) 4, 14, 24, 34		4, 14, 24, 34, __, __, __
ii) −4, −2, 0, 2	$-2 - (-4) = 0 - (-2) = 2$	−4, −2, 0, 2, __, __, __
iii) $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$		$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$, __, __, __

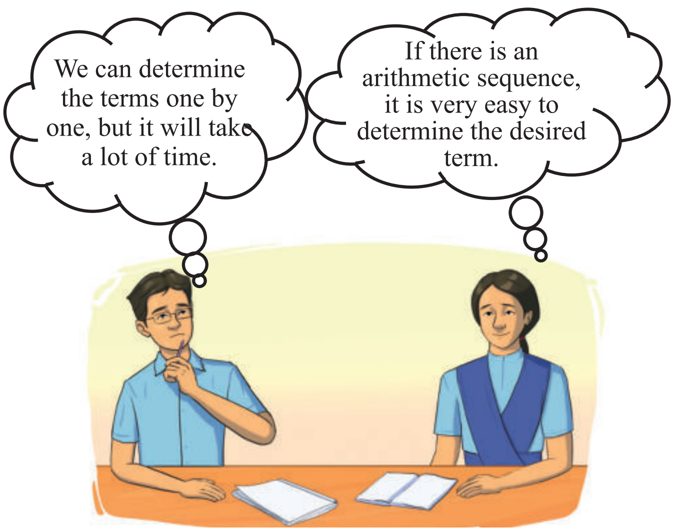
You can easily find the next three terms, can't you? But what if you were asked to determine the 120th or 350th term? Would you be able to calculate it one by one like this? Find about if the task would still be easy or not.

So, let us s see if we can find a common term or determine the nth term of any sequence.

Let's take an example: 3,8,13,18,………

Certainly, this is an arithmetic sequence, isn't it?

Now, let's analyze step by step and investigate what kind of characteristics are



present among the terms in the sequence. We'll examine them one by one and see if we can find any common pattern:

Term	Sequences given	Pattern	We can write	General term or n-th term
1	3	3	$3+5(0)$	$a_n = 3 + (n-1)\ 5$; a_n: n^{th} term; 3: 1st term a_1; 5: Common difference, d
2	8	3+5	$3+5(1)$	n^{th} term $a_n = a_1 + (n-1)d$
3	13	3+5+5	$3+5(2)$	
4	18	3+5+5+5	$3+5(3)$	
...	...	...	...	
n	a_n	3+5+5+5+5+ ⋯ +5	$3+5(n-1)$	

Example - 7: Determine the 15th, 120th, and the general term (n-th term) of the following sequence in order: 7,11,15,19,……

Solution: The given sequence 7, 11, 15, 19,…… is an arithmetic sequence because

2nd term – 1st term = 11 – 7 = 4,

3rd term – 2nd term = 15 – 11 = 4,……

It means the common difference of the sequences is 4

We know, nth term

$$a_n = a_1 + (n-1)d$$

where, 1st term = $a_1 = 7$, no. of terms = n and common difference = $d = 4$

$\therefore$ nth term $a_n = 7 + (n-1)4 = 7 + 4n - 4 = 4n + 3$

$\therefore$ 15th term $a_{15} = 4 \times 15 + 3 = 63$

And 120th term $a_{120} = 4 \times 120 + 3 = 483$

Brain storm 

If $7x + 2$, $5x + 12$, $2x - 1$ is an arithmetic sequence, find the value of x.

Individual task:

A) Determine the general term of the following arithmetic sequence:

i) 5, 12, 19, 26,… ii) 1, 0.5, 0, −0.5,… iii) whose 7^{th} term is -1 and 16^{th} term is 17

B) Determine the missing terms in the following arithmetic sequence.

i) 6,______,______,______, 54 . ii) ______, −3, 2, ______,______, 17.

Group Work

Take one number from each of the boxes below, and make five arithmetic sequences.

2	-9	4	-3	-16
0	24	15	8	10
5	6	31	10	-12
3	20	-6	-4	10
-12	-20	-8	17	25

Geometric Sequence

Let's shed light on two events.

Event - 1

Lily wants to buy a gift for her mother. She needs at least 300 taka to buy the gift. Lily plans to save some money starting from the first week of November. The amount she saves in the first week will be doubled the following week. According to the plan, Lily started saving with 5 taka. In how many weeks Lily will be able to buy the gift?

Now, let's calculate it:"

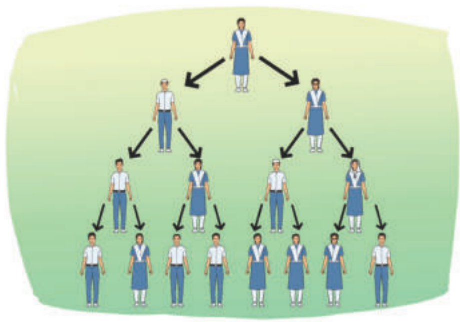
No. of weeks	1	2	3	4	5	6	Total savings
Weekly savings (Taka)	5	5× 2	10 × 2	20 × 2	40 × 2	80 × 2	315
		10	20	40	80	160	

Since Lily will have a total of 315 taka saved, she will be able to buy the gift for her mother at the end of the sixth week.

You must have noticed that Lily's savings plan has a special feature, mustn't you? Let's take a closer look.

$$\frac{10}{5} = 2, \quad \frac{20}{10} = 2, \quad \frac{40}{20} = 2, \quad \frac{80}{40} = 2, \quad \frac{160}{80} = 2$$

Event – 2: Spreading of Virus



We are often affected by various diseases caused by viruses. Let's assume that a virus-borne disease spreads in such a way that initially one person gets infected. Then it continues to spread like a pattern. The sequence for how the disease spreads can be described as: 1, 2, 4, 8, ... where each term is double the previous term except for the first term.

What did we find by analyzing both events?

In both cases, each term is double the previous term. Alternatively, the ratio between any term and its preceding term is a specific constant number. In other words, the ratio of any term to its previous term is constant. This type of sequence is called a **geometric sequence.**

Example: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is geometric sequence because the ratio of any term to its previous term is $\frac{1}{2}$.

Individual work:

Write two geometric sequences in box below where the common ratio of one sequence will be $\frac{1}{3}$ and you choose by your own the common ratio of the other.

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In a geometric sequence, the first term is denoted as a, and the common ratio is denoted as r. Therefore, we can write a geometric sequence in the following form:

$$a, ar, ar^2, ar^3, \ldots$$

1st term of the sequence is a and the common ratio is r because -

2nd term ÷ 1st term = $ar \div a = r$

3rd term ÷ 2nd term = $ar^2 \div ar = r$

4th term ÷ 3rd term = $ar^3 \div ar^2 = r$

Can you tell me what will be the n^{th} term of the sequence? Observe that the n^{th} term will be,

$$a_n = ar^{n-1}$$

- Give two examples of finite and infinite geometric sequences.
- If a, b, c are in geometric progression, fill in the blank spaces below:

$$\frac{\square}{\square} = \frac{\square}{\square}$$

Determining the common term or n^{th} term of geometric sequence

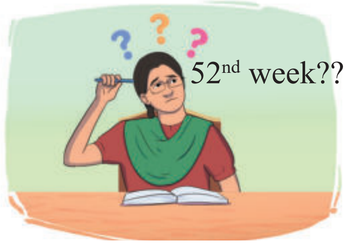
Find the common ratio and the next three terms of the following sequences. You are given one ratio as an example:

Sequence	Common ratio	Sequence including the next three terms
i) 6, 18, 54,…	$18 \div 6 = 54 \div 18 = 3$	6, 18, 54, <u>162</u>, <u>486</u>, <u>1458</u>, ...
ii) $3x, 9x^2, 81x^3, \ldots$		$3x, 9x^2, 81x^3$, __, __, __, ...
iii) 625, 25, 5,…		625, 25, 5, __, __, __, ...
You could certainly find the next three terms of the sequence quite easily, couldn't you? However, if you are asked to find the 50^{th} or 100^{th} term, will do it in the same way? Think if it will be a feasible way to do the work or not.		

Pair work:

Just like an arithmetic sequence, analyze each term using an example. Investigate if any common characteristics can be found among the terms.

Lily's weekly savings sequence:



Consider Lily's weekly savings sequence:

$$5, 10, 20, 40, \ldots$$

Now, if Lily is asked how much money she needs to save by the end of one year or in the 52nd week?

Term	Sequences given	Pattern	We can write	Common term or n^{th} term
1	5	5	5×2^0	$a_n = 5 \times 2^{n-1}$; n^{th} term, 1st term a_1, Common difference, d ; n^{th} term $a_n = a_1 r^{n-1}$
2	10	5×2	5×2^1	
3	20	$5 \times 2 \times 2$	5×2^2	
4	40	$5 \times 2 \times 2 \times 2$	5×2^3	
...	...	...	...	
n	a_n	$5 \times 2 \times 2 \times 2 \times \ldots \times 2$	$5 \times 2^{n-1}$	

Now, think about it, if Lily had known this simple concept of finding the common term or the n^{th} term of a geometric sequence, would it be difficult or time-consuming for her to determine how much money she would have saved in the 52nd week?

Quickly calculate and take decision:

Example

Let's say you have a job that allows you to study simultaneously. By doing this job, you earned 4000 taka in the first month. Each subsequent month, your income increases by 5% compared to the previous month. How much will your income be in the 10th month?

Solution: Let's analyze this problem using a chart:

Month	1^{st}	2^{nd}	3^{rd}	...	10^{th}
Income (Taka)	4000	4200	4410	...	?

First, we need to determine whether the given sequence is an arithmetic or geometric progression. You can do the investigation yourself.

1^{st} term of sequence $a_1 = 4000$, common ratio $r = 1.05$ and no. of terms $n = 10$

You have already learnt the formula of determining the general term or nth term of geometric series. $= a_1 r^{n-1}$

$\therefore$ In 10^{th} month your income will be $= 4000\ (1.05)^{10-1} = 4000\ (1.05)^9 = 6205.31$ taka (approx..)

Brain storm

If $x + 6$, $x + 12$, $x + 15$ form a geometric sequence, determine the value of x.

Instructions

You have already known if a, b, c are consecutive terms in a geometric sequence, $\frac{b}{a} = \frac{c}{b}$ or $b^2 = ac$

Individual Task:

A) Determine the general terme of the following arithmetic sequence.

i) 3, 15, 75, … ii) 4, $\frac{4}{5}$, $\frac{4}{25}$, … iii) whose 7^{th} term is 8 and 13^{th} term 512

B) Determine the missing terms in the following geometric sequence.

i) 3,_____,_____,_____, $\frac{1}{27}$ ii) _____, $\frac{1}{8}$, _____,_____, $\frac{1}{64}$

C) i) In the sequence 4, 12, 36, ..., what is the position (term number) of 2916??

ii) In a geometric sequence, if $a_4 = \frac{8}{9}$ and $a_7 = \frac{64}{243}$, what is a_{10}?

Fibonacci Sequence

Fibonacci pattern is also a sequence but this is different from arithmetic and geometric sequences. Look at the following sequences. Can you identify the next term for each of the following sequences? If you can, write them in the following table with logic.

Sequence	Next term	Logic
(i) 3, 5, 7, 9, 11,…		
(ii) 3, 6, 12, 24, 48,…		
(iii) 3, 5, 8, 13, 21,…		

You can easily identify the next term in the sequence number (i) and (ii), can't you? The reason is that you have an idea about the characteristics of two side by side terms. The next term in the sequence number (iii) cannot be identified. In this sequence, the terms of the sequence do not repeat the same characteristics. Since it is called a sequence, there must be some repetition of the same characteristics among its terms. It is a special kind of sequence.

Italian mathematician named Leonardo Pisano discovered the beauty of the relationship among the terms of this special sequence. His nickname was Fibonacci. By searching nature, he discovered this special sequence which was published in a book called "Liber Abaci".

Leonardo Pisano (Fibonacci)

In the Fibonacci sequence of numbers where the first two numbers are 0 and 1. In the Fibonacci sequence, any term is equal to the sum of the previous two terms. So, the 3rd term = 0 + 1 = 1, the 4th term = 1 + 1 = 2. Then the Fibonacci sequence would be-

0, 1, 1, 2, 3, 5, 8, 13, 21,……..

We can find the terms of the Fibonacci sequence from the formula below:

$$F_{n+2} = F_{n+1} + F_n \ ; n \in N$$

where, $F_0 = 0$, $F_1 = F_2 = 1$

Fibonacci sequence building game, **Time: 5 Minute**

Fibonacci sequences have to be formed within the given time. Whoever can form the maximum number of terms of the Fibonacci sequences will win.

Fibonacci Sequences in Nature

Fibonacci was a nature loving mathematician. He used to research various mysteries of nature's creation. While researching this mystery of nature's creation he found that nature has always followed a regular pattern in its creation.

Observe the pictures below.

Note that the number of petals in each flower follows the Fibonacci sequence.

Individual/ group work:

Make a report by observing the plants in your school garden or around the house whose branches, number of leaves or number of flower petals are similar to the Fibonacci sequence.

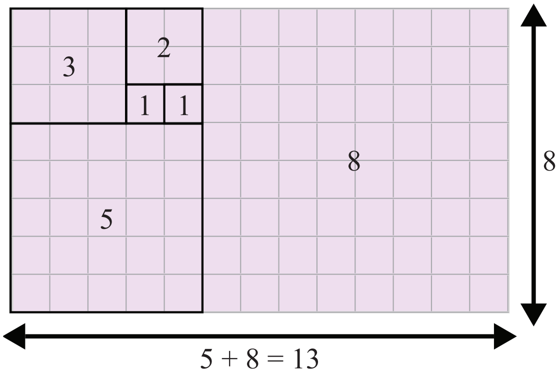
Fibonacci Rectangle

Look at the adjacent picture. It is a rectangle. If the length of the side of its smallest squares is 1 unit, the area of the rectangle can be identified by counting the smaller squares of the image, right?

So, quickly tell the area of the rectangle

= $\boxed{}$ square unit.

The area of the rectangle can be measured in different ways.



In the picture, the length of the rectangle = 13 unit and the width = 8 unit.

Therefore, the area = (13 × 8) = 104 square unit.

This is a geometric calculation of a rectangle. If you look closely, you will find some numbers inside the rectangle. If you chronologically arrange the numbers from the smallest one, you will get a sequence. You can surely recognize the sequence. Now if we draw a square on top of each other according to the terms of the Fibonacci sequence, and draw in this way no matter where we stop, we will always see a rectangle. This rectangle is called the Fibonacci rectangle. Again, the squares of different shapes will be formed inside the rectangle, the sum of their areas will be the area of the rectangle. Let us check it.

∴ The area of the rectangle = $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 104 = (13 \times 8)$ square unit.

Individual work:

Draw the Fibonacci rectangle using numbers 1, 1, 2, 3, 5, 8, 13, 21 on graph paper or grid.

Series

You already know about sequences. If the terms of the sequence are joined by (+) sign, the series is obtained. For example-

Example 1. Series of constant numbers: $3 + 3 + 3 + 3 + \cdots$

Example 2. Series of natural numbers: $1 + 2 + 3 + 4 + \cdots$

General Form of Series:

If the 1st term is a_1, the 2nd term is a_2, the 3rd term is a_3, ... , and n^{th} term is a_n in a sequence, we can write the sequence as $a_1, a_2, a_3, \ldots, a_n, \ldots$. The terms of this sequence are joined by (+) sign. It means that $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is a series.

It is called **general form of series**

Finite and Infinite Series

If the number of terms of the series is definite, it is called Finite series and if the number of terms is indefinite, it is termed as Infinite series. For example –

$1 + 3 + 5 + 7 + \cdots + 61$ is a finite series

and

$1 + 3 + 5 + 7 + \cdots$ is an infinite series.

Individual work

Give examples of two finite and infinite series.

Two important series

Similar to sequences, there are two types of clauses depending on the relationship between two consecutive terms. For example –

(i) $1 + 3 + 5 + 7 + \cdots$

and

(ii) $2 + 4 + 8 + 16 + \cdots$

are two series. The difference between two consecutive terms of the first one is equal. Again the ratio of the two consecutive terms of the second one is equal.

Arithmetic Series

Same thing goes for arithmetic series. If the 1st term is a_1, the common difference is d and n^{th} term is a_n in an arithmetic series, the series would be-

$$a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + \{a + (n - 1)d\} + \cdots$$

where the n^{th} term if the series

$$a_n = a_1 + (n - 1)d \quad [\text{considering } a = a_1]$$

Individual task:

a) Find out the n^{th} term of the following arithmetic series.

i) $6 + 13 + 20 + \cdots$ ii) $2 - 5 - 12 - \ldots$

b) Find out the blank terms of the following arithmetic series.

i) $8 - 8 - 24 - \_ - \_ - \_$

ii) $\_ -3 + 2 + \_ + \_ + 17.$

Brain storm

For which value of k would $(5k - 3) + (k + 2) + (3k - 11)$ be a arithmetic series? Find out the common difference and the common term of the series.

Sum of arithmetic series

As you know, Bangladesh is often affected by cyclones due to its geographical location. When there is a cyclone, our plants suffer the most. Houses, roads, and hundreds of trees are damaged. You also know how much these green plants benefit us. So we all need to plant trees for our protection. In this case, you can take a five-year plan for planting trees in your area by talking to the head of the institution through your class teacher. The plan can be planting 10 fruit trees in the 1^{st} month and later on planting 5 more trees every next month compared to the previous month. Just imagine how many trees you can plant in five years.

Let's calculate the first year (12 months). Some details are given in the table below, you fill in the rest:

Number of month	1	2	3	4	5	6	7	8	9	10	11	12	Total
Number of trees	10	15	20									65	??

You can understand that if you convert the calculation into a series, it would be an arithmetic series. Arithmetic series is:

$$10 + 15 + 20 + \cdots + 65$$

Think about it. Moreover, the number of trees required for a month may be calculated using the formula of identifying the common term. But will it be easy to figure out how many trees you can plant in one year? Let us discuss how to calculate the total number.

Instruction

n^{th} term of arithmetic series is

$$a_n = a + (n - 1)d$$

One of the best mathematicians, Carl Friedrich Gauss, wrote the basis for determining the sum of the first series. We will discuss here Carl Gauss's method for determining the sum of series.

Carl Friedrich Gauss

Let us observe how Carl Gauss found the sum of the first n number of natural numbers

Let, the series of the first n natural numbers

$$S_n = 1 + 2 + 3 + \cdots + n$$

The series can be written in reverse, $S_n = n + (n - 1) + (n - 2) + \cdots + 1$

By adding two series, we get, $2S_n = \underbrace{(n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1)}_{n \text{ times}} = n(n + 1)$

$$\therefore S_n = \frac{n\,(n + 1}{2}$$

This is Gauss's formula for sums of first n-times natural numbers.

Individual work

Determine the sum of the first 50 natural numbers using Gauss's formula.

Now let us return to the problem of planting trees. The summation series for the number of trees planted in a year was,

$$S = 10 + 15 + 20 + \cdots + 65$$

The series can be written in reverse, $S = 65 + 60 + 55 + \cdots + 10$

By adding two series, we get, $2S = \underbrace{75 + 75 + 75 + \cdots + 75}_{12 \text{ times}} = 12 \times 75$

$$\therefore S = \frac{12 \times 75}{2} = 6 \times 75 = 450$$

That means, in one year 450 trees can be planted.

Formula for determining the sum of arithmetic series

Let the first term of a arithmetic series be a and the common difference d. Then the n^{th} term $a_n = a + (n - 1)d$

Let S_n be the sum of the first n terms, then

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

The series can be written in reverse,

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + (a + (n - 3)d) + \cdots + a$$

By adding two series, we get

$$2S_n = \underbrace{\{2a + (n - 1)d\} + \{2a + (n - 1)d\} + \cdots + \{2a + (n - 1)d\}}_{n \text{ times}}$$

$$= n\{2a + (n - 1)d\}$$

$$\therefore S_n = \frac{1}{2} n\{2a + (n - 1)d\}$$

This is the formula for determining the sum of arithmetic series of n – terms.

Calculate how many trees you can plant in five years in the blank box below:

Pair Work:

1. Suppose, you have got a job. You have saved 1200 taka from your salary in the 1st month. You will save 100 taka more every next month compared to the previous month.

   a) How much will you save in 20th month?

   b) How much will you save in the 1st 20 months?

   c) In how many years can you save a total of 106200 taka?

2) If $a_8 = 60$ and $a_{12} = 48$, calculate the value of a_{40} and S_{40}.

3) The first and the last terms of an arithmetic series are 3 and −53 respectively. If the sum of the series is −375, how many terms are there?

Geometric Series

You must have remembered that Lily saved money to buy a gift for her mother. Apu like Lily saves money in that bank every week. He has a clay bank in which he puts 2 rupees in the first week, 4 rupees in the second week, 8 rupees in the third week and so on every week twice the amount of the previous week. After three months, Apu becomes very happy holding the bank. Because its weight has increased more than before and by shaking it close to the ear, there is quite a sound of money moving inside. Apu wants to calculate the amount inside. So, he quickly sits down with a pen and paper.

At first, Apu makes the following table like Lily did.

Three months is $\frac{1}{4}$ part of one year. So, 3 months = $\left(52 \times \frac{1}{4}\right)$ = 13 weeks.

The 13 weeks schedule is given below. Fill in the blanks.

Number of weeks	1	2	3	4	…	11	12	13	Total
Amount of money	2	4	8					8192	??

Apu already knows about geometric sequences and their properties. So, he considers the weekly deposited money as a term and creates a series. The series is:

$$2 + 4 + 8 + 16 + \ldots + 8192$$

The first term of the series is $a_1 = 2$, the common ratio is $r = \frac{4}{2} = 2$.

Apu analyzes the series as follows:

Term (Week)	Amount of money	Pattern	Algebraic form	
1	2	$2 \times 1 = 2 \times 2^0 = 2 \times 2^{1-1}$	ar^{1-1}	a
2	4	$2 \times 2 = 2 \times 2^1 = 2 \times 2^{2-1}$	ar^{2-1}	ar
3	8	$2 \times 2 \times 2 = 2 \times 2^2 = 2 \times 2^{3-1}$	ar^{3-1}	ar^2
4	16	$2 \times 2 \times 2 \times 2 = 2 \times 2^3 = 2 \times 2^{4-1}$	ar^{4-1}	ar^3
…	…	…	…	…
n		$\underbrace{2 \times 2 \times 2 \times 2 \times \ldots \times 2}_{n \text{ terms}} = 2 \times 2^{n-1}$	ar^{n-1}	ar^{n-1}

So, the finite geometric series will be, $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Individual Task:

a) Determine the geometric series: i) $a = 4, r = 10$ ii) $a = 9, r = \frac{1}{3}$ iii) $a = \frac{1}{\sqrt{2}}, r = -\sqrt{2}$

b) How much money did Apu deposit in the bank in the 12th month?

Determining the Sum of the first *n* number of terms of a Geometric Series

Apu plans to find out how much money has been deposited in the clay bank in three months without breaking it. He has two ideas in his mind. The first one is - to determine the total amount of money deposited in 12 weeks by calculating the money for each week and adding them up. But this is a very time-consuming process. And the second one is - to find a way to directly know the total amount of money. Apu's inquisitive mind settles on the second one.

If a is the first term, r is the common ratio ($r \neq 1$) and if the sum of the first n terms of the geometric series is S_n, we can write-

$$s_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \quad \ldots\ldots\ldots\ldots(1)$$

and multiplying both sides by r, $rs_n = \quad ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$ ……(2)

Now subtracting (2) from (1) we get,,

$$S_n - rS_n = a - ar^n$$

$$\text{or, } S_n(1 - r) = a(1 - r^n)$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$$

This is the addition formula for geometric series.

Do you remember which sequence Apu got by saving money for 13 weeks? Yes, the sequence was

$$2 + 4 + 8 + 16 + ... + 8192$$

The first term of the series is $a_1 = 2$,, the common ratio is $r = 2$ and the number of terms is n =12. Substitute the values in the formula and find out the total amount of money Apu has saved in three months.

Individual Task:

a) Find the sum of 7 terms of the series $1 - 3 + 9 - 27 + \cdots$

b) Find the sum of the series $54 + 18 + 6 + \cdots + \frac{2}{81}$

Determining the Sum of infinite terms of a Geometric Series

If the first term is a and the common ratio is $r \neq 1$ of a geometric series, then the sum of the first n terms of the series is

$$s_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

$$= \frac{a(1 - r^n)}{1 - r}$$

But if the number of terms of the series is infinite, do we have to think differently to determine its sum? Let's investigate.

Tell when the series will be infinite? If the value of n is infinite, the sequence will also be infinite. In this case we can write,

$$s_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

Now tell, what will be the sum of this series? The sum of the terms depends on the value of r.

Think about it,

a) When $|r| < 1$ i.e., $-1 < r < 1$

If the number of terms n of the series increases ($n \to \infty$), then the value of $|r^n|$ decreases gradually and becomes close to 0. As a result, the sum of the series .

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

can be written as,

$$S_\infty = \frac{a}{1 - r}$$

b) When $|r| > 1$ i.e., $r > 1$ or $r < -1$

If the number of terms n of the sequence increases ($n \to \infty$), then the value of $|r^n|$ increases gradually and there is no specific value. Therefore, in this case, it is not possible to find the sum of the series.

$-1 < r < 1$

if $r = \frac{1}{2}$, $r^n = ??$

$r^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$,

$r^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$,

$r^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, …

So it is seen that as the value of n increases, r^n decreases.

Individual Task:

1. Make infinite geometrical series in each case.

i) $a = 4, r = \frac{1}{2}$ ii) $a = 2, r = -\frac{1}{3}$

iii) $a = \frac{1}{3}, r = 3$ iv) $a = 1, r = -\frac{2}{7}$

Problem

Suppose you have planted a sapling near your house or in your garden. After one year the height of the sapling is 1 meter. The following year its height increases by 0.8 m. Each year the height of the tree increases by 80% of the previous year's growth height. If it grows like this, how many meters can be the maximum height of the tree?

Solution:

The height of the tree in the first year was = 1 m.

In the second year the height of the tree increased = 0.8 m.

In the third year the height of the tree increased = $(0.8 \times 80\%) = 0.64$ m.

In the fourth year the height of the tree increased = $(0.64 \times 80\%) = 0 .512$ m.

Thus, the height of the tree continued to increase every year. Let's see if we can find any series in the growth of the tree every year.

The series of growth of the tree will be $= 1 + 0.8 + 0.64 + 0.512 + \cdots$

Here, the first term of the series $a_1 = 1$ and the common ratio $r = \frac{0.8}{1} = \frac{0.64}{0.8} =$ 0.8

Since the series is an infinite geometrical series and $-1 < r < 1$

So, the sum of the series.

$$s_\infty = \frac{a}{1-r} = \frac{1}{1-0.8} = \frac{1}{0.2} = 5 \text{ m.}$$

So, the maximum height of the tree will be = 5 m.

Exercise

1. Are the following sequences arithmetic, geometric, Fibonacci or none of the three? Why? Determine the common terms and explain.

   (i) 2, 5, 10, 17,…… (ii) –2, 7, 12, 17,…… (iii) –12, 24, –48, 96,……

   (iv) 13, 21, 34, 55,…… (v) 5, –3, $\frac{9}{5}$, $-\frac{27}{25}$,…… (vi) $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}$,……

2. Fill in the gaps in the following sequences:

   (i) 2, 9, 16, ____,____, 37,____. (ii) –35, ____, ____, –5, 5, ____.

   (iii) ____,____, ____, 5, –4,____. (iv) ____, $10x^2$, $50x^3$,____, ____,

3. Fill in the blank cells of the following table.

Serial No.	1st Term a_1	Common Difference d	Number of terms n	n^{th} term	S_n
i.	2	5	10		
ii.	–37	4			–180
iii.	29	–4		–23	
iv.		–2	13	10	
v.	$\frac{3}{4}$	$\frac{1}{2}$		$\frac{31}{4}$	
vi.	9	–2			–144
vii.	7		13	35	
viii.		7	25		2000
ix.		$-\frac{3}{4}$	15		$\frac{165}{4}$
x.	2	2			2550

4. You want to make a mosaic in the shape of an equilateral triangle on the floor of your study room, whose side length is 12 feet. The mosaic will have white and blue tiles. Each tile is also an equilateral triangle with a side length of 12 inches. The tiles have to be placed in opposite colors to complete the mosaic.

   a) Make a model of triangular mosaic.

   a) How many tiles of each color will you need?

   b) How many tiles will you need in total?

5. Fill in the blank cells of the following table:

Serial No.	1^{st} Term a	Common Ratio r	Number of terms n	n^{th} term	S_n
i.	128	$\frac{1}{2}$		$\frac{1}{2}$	
ii.		-3	8	-2187	
iii.	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$		$8\sqrt{2}$	
iv.		-2	7	128	
v.	2	2			254
vi.	12			768	1524
vii.	27	$\frac{1}{3}$		$\frac{1}{3}$	
viii.		4	6		4095

6.

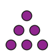
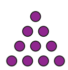
Figure No	Figure	Number of coins
1		1
2		3
3		6
4		10
.....	....	....

Figure-1

n	Number in the row	Sum of numbers in the row
1	1 1	$1 + 1 = 2$
2	1 2 1	$1 + 2 + 1 = 4$
3	1 3 3 1	$1 + 3 + 3 + 1 = 8$
4	1 4 6 4 1	$1 + 4 + 6 + 4 + 1 = 16$
.....	... ...	... ...

Figure-2

a) Observe the sequence in Figure - 1 closely. Then construct the 10^{th} figure and determine the number of coins.

b) Determine the number of coins in the nth figure based on the given information.

c) If $n = 5$, determine the numbers in the 2^{nd} column of Figure - 2 and show that the sum of the numbers in the n^{th} row supports the formula 2^n.

d) Create a series with the sums of each row and determine the value of n when the sum of the first n terms of the series is 2046.

Academic year 2024

7. Determine the value of n, where n ∈ N.

i. $\sum_{k=1}^{n}(20 - 4k) = -20$

ii. $\sum_{k=1}^{n}(3k + 2) = 1105$

iii. $\sum_{k=1}^{n}(-8).\,(0.5)^{k-1} = -\frac{255}{16}$

iv. $\sum_{k=1}^{n}(3)^{k-1} = 3280$

8. The first, second and tenth terms of an arithmetic series are equal to the first, fourth and seventeenth terms of a geometric series respectively.

a) If the first term of the arithmetic series is '*a*', the common difference is '*d* 'and the common ratio of the geometric series is '*r*', then form two equations by combining the two series.

b) Determine the value of common ratio '*r*'.

c) If the tenth term of the geometric series is 5120, then determine the values of '*a*' and '*d*'.

d) Determine the sum of the first 20 terms of the arithmetic series.

9. Draw an equilateral triangle. Draw another equilateral triangle connecting the midpoints of its sides. Draw another equilateral triangle connecting the midpoints of the sides of that triangle. Draw 10 triangles in this manner. If each side of the innermost triangle is 64 mm, find the sum of the perimeters of all the triangles.

10. Shahana plunted a sapling in his educational institute. After one year, the height of the sapling is 1.5 feet. The next year, its height increases by 0.75 foot. Every year, the height of the tree increases by 50% of the previous year's increase. If it continues to grow like this, how high can the tree be after 20 years?

11. Find out your family's expenses for the last six months. Consider each month's expenses as a term and try to convert them into a series if possible. Then try to solve the following problems.

a) Is it possible to create a series? If so, explain what kind of series you have got.

b) Express the sum of the series in an equation.

c) Determine how much can be the total expenses for the next six months.

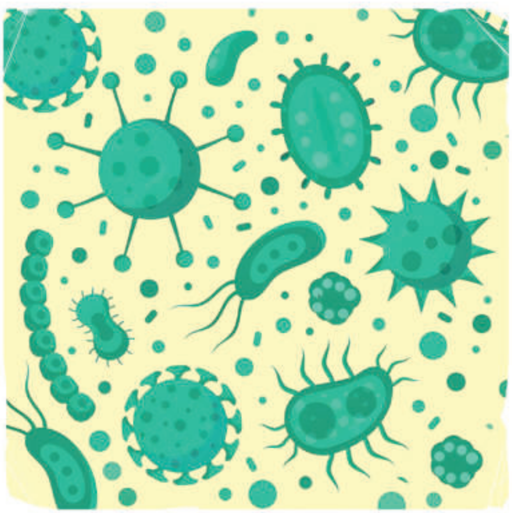
d) Write down your observations about your family's monthly/annual expenses.

Academic year 2024

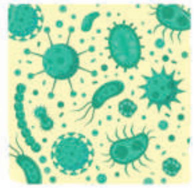
Concept and Application of Logarithm

You can learn from this experience-

- Characteristics of exponent
- Concept of logarithm
- Relationship between exponent and logarithm
- Base of logarithm and their limitations
- Arguments for logarithm and their limitations
- Laws of logarithm and their proofs
- Characteristics of logarithm
- Application of logarithm



Concept and Application of Logarithm



Do you know that bacteria reproduce at a very fast rate? imagine a scenario where the number of bacteria detected in an examination in a particular environment is 4500. These bacteria double in number every hour. you can certainly understand that within a few hours the number of bacteria will increase significantly. For example-

The bacterial count in the 1st hour can be expressed as- $= 4500 \times 2$
$= 9 \times 1000 = 9 \times 10^3$

The bacterial count in the 2nd hour can be expressed as- $= 9000 \times 2$
$= 1.8 \times 10000 = 1.8 \times 10^4$

You know that these kinds of forms are called exponents. You can understand that we can express very large numbers using exponents.

Calculate the bacterial growth from the 1st to the 10th hour and complete the table 3.1.

Individual work- 01

Table - 3.1

Time	Number of bacteria	expressed in exponents
1st hour	$4500 \times 2 = 9000$	4500×2^1
2nd hour	$9000 \times 2 = 18000$	4500×2^2
3rd hour		
4th hour		
5th hour		
6th hour		
7th hour		
8th hour		
9th hour		
10th hour		

So, in the above calculations, we can see that large numbers can be expressed in exponents.

Can you write down the number of bacteria at the n^{th} hour? At the n^{th} hour, the number of the bacteria will be 4500×2^n. if the number of bacteria at the n-th hour is 147,456,000, we can write $4500 \times 2^n = 147,456,000$. This kind of equation is called the **exponential**

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equation. Now tell us how we can get the value of n from this equation. How can we find out the value of an unknown exponential expression from an exponential equation? The usual form of an exponential equation is $b^n = a$ where $b > 0$ and $b \neq 1$. Now the question is how we can find out the value of n?

In this case we can use logarithm. Using Logarithm exponential equation can be writen as That means,

$b^n = a \Leftrightarrow \log_b a = n,$ that is, n is logarithm of a with base b.

$b^n = a$ (where $b > 0$ and $b \neq 1$) if and only if $n = log_b a$


John Napier

b is called the base of log. log is the short form of logarithm. A question must be raised in your mind. Who first gave the idea oflog. Let us briefly know about log. The word logarithm derived from the Greek words logos and arithmos. logos means ratio and arithmos means number. So, the word logarithm means the ratio of numbers. Scottish mathematician John Napier first used this term in one of his books in 1614. You already know about exponents or exponential expressions in detail. Actually, with a view to finding out the value of exponential expression, logarithm is used.

Exponent and log are the same idea but they can be expressed in two different ways. We can express a number through an exponent. We can express the same number through log. It does not change the value of the number. For instance, 9 can be expressed as 3^2. Then $3^2 = 9$ is an equality of exponential expression. Can you tell us what other way we can write exponent 2. 2 is the 3 based log of 9. If we mathematically express the statement, it would be 9. Likewise, from the exponential relationship $2^3 = 8$, we can say that 3 is the 2 based log of 8. If we mathematically express the statement, it would be $3 = log_2 8$.

Notice that in case of exponential equality $2^3 = 8$, the base of exponent is 2. Again in case of $3 = log_2 8$, the base of log is 2.

Therefore, the base of the exponent and the base of the log are the same or equal.

Pair work

The relationship between exponential equality and *log* has been shown in the following table ((Table- 3.2). Fill in the blanks.

Table- 3.2		
Exponential equality	**Expressed in *log***	**Mathematically expressed through *log***
$3^2 = 9$	Exponent 2 is the 3 based *log* of 9	$2 = \log_3 9$
$2^3 = 8$		
$7^{\frac{1}{2}} = \sqrt{7}$		
$2^{-6} = \frac{1}{64}$		$-6 = \log_2 \frac{1}{64}$
	Exponent -4 is 3 based log of $\frac{1}{81}$	

We have so far learned the mathematical presentation of the relationship of exponents by expressing in *log*. Now validate your experience by filling the following table expressing the relationship of exponents in *log* and expressing the relationship of *log* in exponents.

Brain storm Pair work

Express the relation of exponent in log.	
Relation of exponent	Relation of log
$2^4 = 16$	$4 = \log_2 16$
$3^4 = 81$	
$2^7 = 128$	
$2^{-3} = \frac{1}{8}$	
$5^{\frac{1}{2}} = \sqrt{5}$	
$3^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}$	
$10^2 = 100$	
$10^5 = 100000$	

Express the relation of log in exponent.	
Relation of *log*	Relation of exponent
$3 = \log_2 8$	$2^3 = 8$
$6 = \log_2 64$	
$3 = \log_3 27$	
$4 = \log_{10} 10000$	
$-4 = \log_2 \left(\frac{1}{16}\right)$	
$\frac{1}{2} = \log_2 \sqrt{2}$	
$3 = \log_{10} 1000$	
$t = \log_b y$	

The Limitation of the Base of log

You have noticed that while converting the relationship of exponent into log, a condition has been imposed on the base b. The condition is $b > 0$ and $b \neq 1$. This the limitation of the base of *log*.

Condition-1: When $b < 0$.

We know that $(-3)^{\frac{1}{2}} = \sqrt{-3}$, and it is unreal. From this relation, we get $\frac{1}{2} = \log_{-3}\sqrt{-3}$.

As we get the unreal value $\sqrt{-3}$ for the base -3, negative number as the base of the log is not acceptable. So, the base of *log* cannot be a negative number.

Condition-2: When $b = 0$.

We know that 0 is for $0^2 = 0$ so, $2 = log_0 0$ and 0 is for $0^3 = 0$ so, $3 = log_0 0$.

Have you noticed one thing? From the above relations we can write $2 = 3$ and it is illogical.

So, $b \neq 0$. That means the base of *log* cannot be 0.

Condition-3: When $b = 1$.

We know that any integer number for n is $1^n = 1$. So, $n = log_1 1$. That means if $n = 4$, $4 = log_1 1 = 0$, and it is illogical. So, $b \neq 1$. That means, the base of *log* cannot be 1.

From the above conditions, we can conclude that-

- the base of *log* cannot be a negative number.
- the base of *log* cannot be 0.
- the base of *log* cannot be 1.

So, we can say that the base of log can be any positive number except 1.

The Arguments for log and Their Limitations

You know that, b is called the base of n. Then what do we call n? n is called the argument of *log*. There are limitations of the argument of *log*.

If $b > 0$ and $b \neq 1$, $y = b^n$ is always positive for any value of n. That is, $b^n = y > 0$ and then $n = log_b y$. That is why the argument of *log* is always a positive number. This is very cautionary information about *log*.

Types of Logarithm

There are two types of logarithm. They are:

- Natural logarithm
- Common logarithm

Natural Logarithm

When the base of the *log* is e, it is called **natural logarithm**. $log_e x$ is expressed in ln x. John Napier first expressed the logarithm considering e as the base. That is why this kind of logarithm is called Napierian logarithm or e based logarithm. e is an irrational number whose value is 2.71828183…

Common Logarithm

An English mathematician Henry Briggs made a *log* table in 1624 considering 10 as the base. This logarithm is called Briggsian logarithm or 10 based logarithm. It is also called common logarithm. Common logarithm is expressed by $log_{10} x$.

You will notice that e is the base of $\ln x$ and 10 is the base of $\log x$. That means-

$$\log_e x = \ln x \text{ and } \log_{10} x = \log x$$

Some Formulas Related to Logarithm

Since the concept of log has come from the concept of exponent, in order to get the formulas of log, we have to know the formulas of the exponent. Fortunately, we already know the formulas of the exponent. For the ease of the work, we have written down the formulas of exponent here.

Formulas of Exponent

For any real number x and y and for any natural number m and n,

1. $x^m \times x^n = x^{m+n}$
2. $\dfrac{x^m}{x^n} = x^{m-n}$ $(x \neq 0)$
3. $(xy)^n = x^n y^n$
4. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$ $(y \neq 0)$
5. $(x^m)^n = x^{mn}$
6. $x^0 = 1$ $(x \neq 0)$
7. $x^{-n} = \dfrac{1}{x^n}$ $(x \neq 0)$
8. $\left(\dfrac{x}{y}\right)^n = \left(\dfrac{y}{x}\right)^{-n}$ $(x \neq 0, y \neq 0)$

Some Formula Related to Logarithm and Their Proofs

Formula 1. $\log_b 1 = 0$

Proof: We know from the exponent that $b^0 = 1$

Expressing this exponential expression in *log* , we get-

$\log_b 1 = 0$ (Proved)

Formula 2. $\log_b b = 1$

Proof: We know from the exponent that $b^1 = b$

Expressing this exponential expression in *log* , we get-

$\log_b b = 1$ (Proved)

Formula 3. $\log_b(AB) = \log_b A + \log_b B$

Proof: Let us imagine, $\log_b A = x$ and $\log_b B = y$

Expressing this logarithmic expression in exponent , we get-

$b^x = A$ and $b^y = B$

or, $b^x b^y = AB$

or, $b^{x+y} = AB$

Expressing this exponential expression in *log* , we get-

$\log_b (AB) = x + y = \log_b A + \log_b B$ [Placing the value of x and y]

$\therefore \log_b (AB) = \log_b A + \log_b B$ (Proved)

Formula 4. $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$

Proof: Let us imagine, $\log_b A = x$ and $\log_b B = y$

Expressing this logarithmic expression in exponent , we get-

$$b^x = A \text{ and } b^y = B$$

$$\therefore \frac{b^x}{b^y} = \frac{A}{B}$$

$$\text{or, } b^{x-y} = \frac{A}{B}$$

Expressing this exponential expression in *log* , we get-

$\log_b \left(\frac{A}{B}\right) = x - y = \log_b A - \log_b B$ [Placing the value of x and y]

$\therefore \log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$ (Proved)

Formula 5. $\log_b A^x = x\log_b A$

Proof: Let us imagine, $\log_b A = y$

Expressing this logarithmic expression in exponent , we get-

$$b^y = A$$

or, $(b^y)^x = A^x$ [Placing the power x on the both sides]

or, $b^{xy} = A^x$

Expressing this exponential expression in *log* , we get-

$\log_b A^x = xy = x \log_b A$ [Placing the value of y]

$\therefore \log_b A^x = x \log_b A$ (Proved)

Formula 6. $\log_a b \times \log_b c = \log_a c$

Proof: Let us imagine, $\log_b c = x$ and $\log_a c = y$

Expressing these two logarithmic relations in exponent, we get-

$b^x = c$ and $a^y = c$

We can write from the above relations-

$b^x = a^y$

By placing the power $\frac{1}{x}$ on the both sides, we get-

$(b^x)^{\frac{1}{x}} = (a^y)^{\frac{1}{x}}$

or, $b = a^{\frac{y}{x}}$

Expressing this exponential expression in *log* , we get-

$\log_a b = \frac{y}{x}$

$\therefore \log_a b \times x = y$

Now, let us place the value of x and y.

$\therefore \log_a b \times \log_b c = \log_a c$ (Proved)

Formula 7. $b^{\log_b a} = a$

Proof: Let us imagine, $\log_b a = x$

Expressing this logarithmic expression in exponent , we get-

$b^x = a$

Now, let us place the value of x.

$\therefore b^{\log_b a} = a$ (Proved)

Formula 8. $x^{\log_b y} = y^{\log_b x}$

Proof: Let us imagine, $y = m$ and $x = n$

Expressing these two logarithmic relations in exponent, we get-

$b^m = y$ and $b^n = x$

Now, $b^m = y$

By placing the power n on the both sides, we get-

$(b^m)^n = y^n$

$\therefore b^{mn} = y^n$ ....(1)

Again, $b^n = x$

By placing the power m on the both sides, we get-

$(b^n)^m = x^m$

$\therefore\ b^{mn} = x^m$ ....(2)

We can write from the relations (1) and (2) that

$x^m = y^n$

Now, let us place the value of m and n.

$\therefore\ x^{\log_b y} = y^{\log_b x}$ (Proved)

Formula 9. $\log_a b = \dfrac{1}{\log_b a}$

Proof: We already know that $\log_a b \times \log_b c = \log_a c$

Now, let us place $c = a$.

$\log_a b \times \log_b a = \log_a a$

or, $\log_a b \times \log_b a = 1$ $[\log_a a = 1]$

$\therefore\ \log_a b = \dfrac{1}{\log_b a}$ (Proved)

Formula 10. $\log_a x = \dfrac{\log_b x}{\log_b a}$

Proof: According to the formula number $\log_a b \times \log_b x = \log_a x$.

$$\Rightarrow \frac{1}{\log_b a} \times \log_b x = \log_a x \qquad [\because \log_a b = \frac{1}{\log_b a}]$$

It means that $\log_a x = \dfrac{\log_b x}{\log_b a}$ (Proved)

Whatever the base is, natural logarithm and common logarithm abide by the following formulas.

1. $log_b 1 = 0$ 2. $log_b b = 1$ 3. $log_b(AB) = \log_b A + \log_b B$ 4. $log_b(\frac{A}{B}) = \log_b A - \log_b B$	5. $log_b A^x = x\log_b A$ 6. $log_a b \times \log_b c = \log_a c$ 7. $b^{log_b a} = a$ 8. $x^{log_b y} = y^{\log_b x}$	9. $log_a b = \frac{1}{\log_b a}$ 10. $log_a x = \frac{\log_b x}{\log_b a}$

Characteristics of Exponents

i. If $b^x = b^y$ for $b > 0$ and $b \neq 1$, $x = y$.

ii. If $a^x = b^x$ for $a > 0$, $b > 0$ and $x \neq 0$, $a = b$.

iii. If $b^x = 1$ for $b > 0$ and $b \neq 1$, $x = 0$.

iv. If $b^x = 1$ for $b > 0$ and $x \neq 0$, $b = 1$.

Characteristics of log

Among many characteristics of log, some notable characteristics have been mentioned below.

1. If $b > 1$ and $x > 1$, $log_b x > 0$.
2. If $0 < b < 1$ and $0 < x < 1$, $log_b x > 0$.
3. If $b > 1$ and $0 < x < 1$, $log_b x < 0$.
4. If $log_b x = log_b y$ for $x > 0$, $y > 0$ and $b \neq 1$, $x = y$.

Let us calculate using log

Example 1. $\log_5 125 = \log_5 5^3 = 3\log_5 5$ [As $\log_a A^x = x\log_a A$]

$= 3 \times 1$ [As $\log_a a = 1$]

$= 3$

Example 2. $\log_c\left(\frac{2\sqrt{40}}{\sqrt{160}}\right) = \log_c\left(\frac{2\sqrt{4 \times 10}}{\sqrt{16 \times 10}}\right) = \log_c\left(\frac{2 \times 2\sqrt{10}}{4\sqrt{10}}\right)$

$= \log_c\left(\frac{4\sqrt{10}}{4\sqrt{10}}\right) = \log_c 1 = 0$ [Since, $\log_c 1 = 0$]

Example 3. $\log_{10}3 + 2\log_{10}5 = \log_{10}3 + \log_{10}5^2$ [As $x\log_a A = \log_a A^x$]

$= \log_{10}3 + \log_{10}25$

$= \log_{10}(3 \times 25)$ [As $\log_a(AB) = \log_a A + \log_a B$]

$= \log_{10}75$

Example 4.

Let us find out the value of x from the relation $\log\left(\frac{1}{49}\right) = -2$

Expressing this logarithmic expression in exponent , we get-

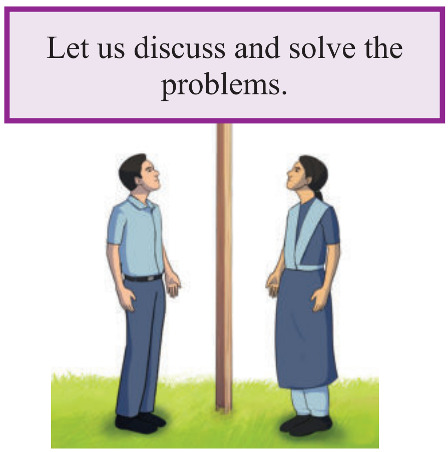
$x^{-2} = \frac{1}{49}$

or, $\frac{1}{x^2} = \frac{1}{49}$

or, $x^2 = 49$

or, $x = \sqrt{49}$ [Excluding the negative value; base x can never be negative]

$\therefore x = 7$



Pair work

Find out the value.

1. $\log_a\left(\frac{\sqrt{140}}{2\sqrt{30}}\right) + \log_a\left(\frac{3\sqrt{12}}{2\sqrt{27}}\right) + \log_a\left(\frac{a^3\sqrt{b^2}}{b\sqrt{a^2}}\right)$

2. $2\log_{10}3 + 3\log_{10}4 + 2\log_{10}5$

The Use of Device in Finding Out the Value of Logarithm

Well, think of how we can find the value of $\log_2 3$. Let us convert log into exponent for better understanding. Let us imagine $\log_2 3 = x$. Then, $2^x = 3$. Now tell us for which value of x, $2^x = 3$? The solution is not simple, is it? For this reason, the log table was made. Nowadays, we can easily calculate these values using various devices like calculators or computers.

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Brain storm Pair work

Complete the following table (3.3) using a device. A few were done up to 5 places after the decimal marker.

Table 3.3					
log	**value**	**exponent**	**log**	**value**	**exponent**
$\log_2 3$	1.58496	$2^{1.58496} \approx 3$	$\log_2 16$		
$\log_3 5$			$\log_5 3$		
$\log_4 7$			$\log_{10} 4$	0.60206	$10^{0.60206} \approx 4$

The Use of Logarithm

There are many uses of logarithm in real life. Some examples have been discussed here.

Logarithm is Compound Interest

You all are familiar with compound interest. Recall the formula of compound interest. It is like-

$$A = P(1 + r)^n$$

Here, P initial capital, A compound capital, r compound interest rate and n number of time period

Problem: In how many years will the compound capital double at 8% interest rate

Solution: Let us imagine, initial capital = P, compound capital $A = 2P$ and the rate of compound interest $r = 8\% = \frac{8}{100} = 0.08$.

So, we get from the formula-

$$2P = P(1 + 0.08)^n$$

$$\text{or, } 2 = (1+0.08)^n$$

$$\text{or, } 2 = (1.08)^n$$

$$\text{or, } n = \log_{1.08} 2 \approx 9$$

So, the capital will double in 9 years.

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Brain storm **Pair work**

In how many years will the compound capital increase by 40% at 12% interest rate?

Logarithm in Calculating the Depreciation of an Object

The decrease in the value of an object over a period of time is called depreciation.

$$P_T = P(1 - R)^T$$

Here, initial price is P, rate of depreciation is R, period of time T and markdown after T is P_T.

Problem: The depreciation of car price

In how much time will the price of a car be reduced to half with a 4% markdown rate?

Solution: We can write from the formula of depreciation-

$$P_T = P(1 - R)^T$$

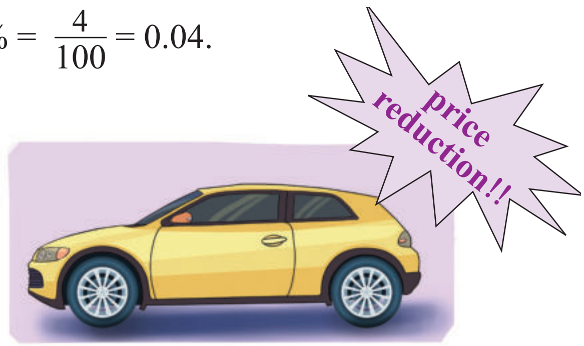
Let us imagine, initial price of the car is P and After the time period of T, the price of the car is reduced to half. That means after the period of T, the price of the car is $P_T = \frac{P}{2}$. The rate of markdown R = 4% = $\frac{4}{100}$ = 0.04.

so,

$$\frac{P}{2} = P(1 - 0.04)^T$$

$$\frac{1}{2} = (1 - 0.04)^T = (0.96)^T$$

$$T = \log_{0.96}(0.5) \approx 17$$



So, the price of the car will be reduced to half in 17 years.

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Brain storm **Pair work: Lifespan of Industrial Machinery**

If the price of a factory's machinery is halved each year, how many years will it take for a 60% price reduction?

Logarithm measuring soil fertility

You know, a good harvest depends on the fertility of the land. As time passes, the fertility of the land decreases. For this reason, to get a good crop, you need to apply fertilizer to the land. The amount of fertilizer to be applied depends on how much the soil fertility has decreased. If we know the rate of decline of soil fertility, we can calculate the required amount of fertilizer. As a result, the wastage of fertilizer will be reduced, the damage to the environment will also be reduced.

Example: If the fertility of the land decreases at the rate of 2% per year, after how many years the fertility of the land will decrease by 30%? If every kg of fertilizer increases the fertility of 1 katha of land by 5%, what amount of fertilizer should be used in 1 bigha of land every year? [1 bigha = 20 katha]

Solution: From the law of depreciation we know, $P_T = P(1 - R)^T$

Here, the initial fertility of the land is P

Rate of fertility decline $R = 2\% = 0.02$.

soil fertility after T time $P_T = P \times (100 - 30)\% = P \times 70\% = 0.70P$

So, $0.70P = P \times (1 - 0.02)^T$

or, $0.70 = (0.98)^T$

or, $T = \log_{0.98}(0.7) \approx 17.6$

So, after 17.6 years the fertility of the land will decrease by 30%.

Again, to increase the fertility of 1 katha land by 5%, required fertilizer is 1 kg.

To increase the fertility of 1 katha land by 2%,required fertilizer is $\frac{2}{5}$ kg

To increase the fertility of 1 bigha land by 2%,required fertilizer is $\frac{2}{5} \times 20 = 8$ kg

[$\because$ 1 bigha =20 katha]

Logarithm in Measuring the Magnitude of earthquakes

We all are familiar with earthquakes. This is a natural disaster. If the magnitude of an earthquake is low, the damage in the area is relatively low. If the magnitude is high, the damage of lives and properties is comparatively high. Scientists can measure the magnitude of earthquakes.

Do you know how to measure the magnitude of an earthquake? Charles Francis Richter proposed the following formula to measure the magnitude of an earthquake.

Magnitude of an earthquake , $R = \log\left(\frac{I}{S}\right)$

Here, I = the maximum intensity covers an area of 100 radius from the source of the earthquake and S = the intensity of a standard earthquake whose amplitude is 1 micron = $\frac{1}{10000}$cm.

The instrument used for measuring earthquake is called seismograph. It was invented by Charles Francis Richter. This instrument was named Richter scale after him. On the Richter scale the magnitude of an earthquake is denoted by R.

In case of a standard earthquake, $I = S$.

Therefore, the magnitude of an standard earthquake is- $R = \log\left(\frac{S}{S}\right) = log 1 = 0$

So, $R = 0$ expresses that no earthquake took place in that place.

An Observation

Let us know a very interesting thing. Can you imagine that a 6 magnitude earthquake is 10 times stronger than a 5 magnitude earthquake? The reason is-

Let us imagine the intensity of a 5 magnitude earthquake is I_5 and a 6 magnitude earthquake is I_6. So,

$$5 = \log_{10}\left(\frac{I_5}{S}\right) \text{ and } 6 = \log_{10}\left(\frac{I_6}{S}\right)$$

$$\therefore \frac{I_5}{S} = 10^5 \quad \text{and} \quad \frac{I_6}{S} = 10^6$$

$$\text{or, } \frac{\frac{I_6}{S}}{\frac{I_5}{S}} = \frac{10^6}{10^5}$$

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or, $\dfrac{I_6}{I_5} = 10$

$\therefore I_6 = 10 \times I_5$

So, we can see that a 6 magnitude earthquake is 10 times stronger than a 5 magnitude earthquake.

Pair work

Show that a 7 magnitude earthquake is 100 times stronger than a 5 magnitude earthquake and a 8 magnitude earthquake is 1000 times stronger than a 5 magnitude earthquake.

An increase in magnitude by 1 on the Richter scale increases the strength of an earthquake 10 times. For the increase of the magnitude by 2 or 3, the strength of earthquake increases 100 or 1000 times. Can you tell why it is happening? Actually, the intensity is measured using a 10 based log.

Problem: In February 2023, an earthquake took place in the southern part of Turkey. The magnitude of that earthquake on Richter scale was 7.8. After 9 hours, another earthquake took place in the southwest of Turkey with a magnitude of 7.5. How many times stronger was the earlier earthquake than the later earthquake?

Solution: Let us imagine,

I_1 = the intensity of the earlier earthquake, I_2 = the intensity of the later earthquake and S = the intensity of a standard earthquake.

So, on Richter scale, the magnitude of the earlier earthquake is = $\log_{10}\left(\dfrac{I_1}{S}\right)$

and the magnitude of the earlier earthquake is = $\log_{10}\left(\dfrac{I_2}{S}\right)$.

According to the question,

$$\log_{10}\left(\frac{I_1}{S}\right) = 7.8 \quad \ldots\ldots(1) \qquad \text{and} \qquad \log_{10}\left(\frac{I_2}{S}\right) = 7.5 \quad \ldots\ldots(2)$$

Subtracting number (2) from number (1), we get-

$$\log_{10}\left(\frac{I_1}{S}\right) - \log_{10}\left(\frac{I_2}{S}\right) = 7.8 - 7.5$$

or, $(\log_{10}I_1 - \log_{10}s) - (\log_{10}I_2 - \log_{10}s) = 0.3$

or, $\log_{10}I_1 - \log_{10}s - \log_{10}I_2 + \log_{10}s = 0.3$

or, $\log_{10}I_1 - \log_{10}I_2 = 0.3$

or, $\log_{10}(\frac{I_1}{I_2}) = 0.3$

Expressing this logarithmic relation in exponent, we get-

$10^{0.3} = \frac{I_1}{I_2}$

or, $\frac{I_1}{I_2} = 10^{0.3}$

or, $\frac{I_1}{I_2} \approx 1.995262315$

$\frac{I_1}{I_2} \approx 2$

$\therefore I_1 \approx 2I_2$

So, the earlier earthquake was two times stronger than the later one.

Group work

Problem 1: On 14 July 1885,an earthquake with a magnitude of 7.0 took place in Manikganj. On 27 July 2003, an earthquake with a magnitude of 5.1 took place at Barkal in Rangamati. How many times stronger was the Manikganj earthquake than the Rangamati earthquake?

Problem 2: At the beginning of the last century, an earthquake in North America was recorded with a magnitude of 8.3 on the Richter scale. Another earthquake took place in South America which was 4 times stronger than the North American one. What was the magnitude of the South American earthquake?

Measuring the Intensity of Sound Using Logarithm

In order to measure the intensity of sound, logarithm is used. Decibel is the unit of the intensity of sound.

The intensity of sound is ,

$$d = 10\log_{10}(\frac{I}{S})$$

Here, I = the maximum intensity of sound in each square meter expressed in watt

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S = the maximum intensity (below which human beings cannot hear) of sound in each square meter expressed in watt

$S = 10^{-12} w/m^2$.

Example1: A sound machine is continuously producing a sound of $2.30 \times 10^2 w/m^2$. In how many decibels will the sound reach the ear of the people present in that place?

Solution: We know that the intensity of sound is $d = 10\log_{10}\left(\frac{I}{S}\right)$.

Here, $I = 2.30 \times 10^2 w/m^2$

and $S = 10^{-12} w/m^2$

$\therefore d = 10\log_{10}\left(\frac{2.30 \times 10^2 w/m^2}{10^{-12} w/m^2}\right)$

$= 10 \log_{10}\left(\frac{2.30 \times 10^2}{10^{-12}}\right)$

$= 10 \log_{10}(2.30 \times 10^{2+12})$

$= 10 \log_{10}(2.30 \times 10^{14})$

$= 10(\log_{10}2.30 + \log_{10}10^{14})$

$= 10(\log_{10}2.30 + 14 \log_{10}10)$

$\approx 10(0.3617278 + 14 \times 1)$

$= 10(0.3617278 + 14)$

$= 10 \times 14.3617278$

$= 143.617278$

≈ 144

$\therefore$ The intensity of sound is 144 decibel.

Pair work

Problem 3: An brick breaking machine is continually producing a sound of $3.14 \times 10^3\ w/m^2$. In how many decibels will the sound reach the ear of the laborers?

Example 2: If the sound level from a source is $4.0 \times 10^{-5} w$ per square meter, what will be the sound level expressed in decibels?

Solution: We know that the intensity of sound is $d = 10 \log_{10}\left(\frac{I}{S}\right)$

Here, $I = 4.0 \times 10^{-5} w/m^2$

and $S = 10^{-12}\ w/m^2$

$$\therefore \text{d} = 10 \log_{10}\left(\frac{4.0 \times 10^{-5} w/m^2}{10^{-12}\ w/m^2}\right)$$

$$= 10 \log_{10}\left(\frac{4.0 \times 10^{-5}}{10^{-12}}\right)$$

$$= 10 \log_{10}(4.0 \times 10^{-5+12}$$

$$= 10 \log_{10}(4 \times 10^{7})$$

$$= 10(\log_{10}4 + \log_{10}10^{7})$$

$$= 10(\log_{10}4 + 7 \log_{10}10)$$

$$\approx 10(0.60206 + 7 \times 1)$$

$$= 10(0.60206 + 7)$$

$$= 10\ (7.60206)$$

$$= 76.0206 \approx 76$$

$\therefore$ The intensity of sound is 76 decibels.

Individual work

Problem 4: An engine running auto rickshaw is producing sound of $2.35 \times 10^{-6} w$ w per square meter. How many decibels will the sound reach in your ear while sitting in that rickshaw?

Example 3: From a hot water pump, a sound of 50 decibels is being emitted. On the other hand, an irrigation pump is emitting a sound of 62 decibels. How much more is the sound intensity of the irrigation pump than that of the hot water pump?

Solution: We know that the intensity of the sound is

$$d = 10 \log_{10}\left(\frac{I}{S}\right), \text{ Here, } d = 50$$

Let us imagine-

in case of hot water pump, the intensity of sound is $I = h$

so, $50 = 10 \log_{10}\left(\frac{h}{S}\right)$

Diving the both sides by 10, we get-

$5 = \log_{10}\left(\frac{h}{S}\right)$

or, $\frac{h}{S} = 10^5$

$\therefore h = 10^5 \times S$………(1)

in case of irrigation pump, the intensity of sound is $I = w$

$\therefore 62 = 10 \log_{10}\left(\frac{w}{S}\right)$

Diving the both sides by 10, we get-

$6.2 = \log10\left(\frac{w}{S}\right)$

or, $\frac{w}{S} = 10^{6.2}$

$\therefore w = 10^{6.2} \times S$………(2)

We get from (1) and (2)-

$\frac{w}{h} = \frac{10^{6.2} \times \mathrm{S}}{10^5 \times \mathrm{S}}$

or, $\frac{w}{h} = 10^{6.2-5}$

or, $\frac{w}{h} = 10^{1.2}$

or, $\frac{w}{h} \approx 15.85$

$\therefore w \approx 15.85 \times h$

So, the sound intensity of the irrigation pump is 15.85 times more than the sound intensity of the hot water pump.

Exercise

1. Find out the value using different formulas.

   (i) $2\sqrt[3]{343} + 2\sqrt[5]{243} - 12\sqrt[6]{64}$ (ii) $\frac{y^{a+b}}{y^{2c}} \times \frac{y^{b+c}}{y^{2a}} \times \frac{y^{c+a}}{y^{2b}}$

2. Using different formulas, prove, $\left(\frac{z^a}{z^b}\right)^{a+b-c} \times \left(\frac{z^b}{z^c}\right)^{b+c-a} \times \left(\frac{z^c}{z^a}\right)^{c+a-b}$

3 Express the following exponential equalities in log and find out the value of x using a digital device.

   (ii) $2^x = 64$ (ii) $(1.2)^x = 100$ (iii) $7^x = 5$ (iv) $(\frac{2}{3})^x = 7$

4. In how many years will the compound capital be 3 times at 10% compound interest rate?

5. You all know the name of corona virus. This virus spreads quickly. If corona virus spreads from 1 person to 3 persons per day, how many people will be infected with corona virus in 1 month? After how many days, 1 crore People will be infected?

6. Setu's uncle has 3 bighas of land. He applies 30 kg of organic fertilizer every year to maintain the fertility of his land. If each kg of fertilizer increases the fertility of the land by 3%, find the depreciation of the land of Setu's uncle? If he does not apply fertilizer to the land, then after how many years will his land have no crops?

7. On 8 July 1918, an earthquake with a magnitude of 7.6 took place at Srimangal in Moulvibazar. Another earthquake with a magnitude of 6.0 took place in Chattogram on 22 November 1997. How many times stronger was the Srimangal earthquake than the Chattogram earthquake?

8. Once, an earthquake with a magnitude of 8 took place in Japan. In the same year, another earthquake took place which was 6 times stronger than the previous one. What was the magnitude of the later earthquake on the Richter scale?

9. In July 1999, an earthquake with a magnitude of 5.2 took place at Maheshkhali in Cox's Bazar. On 6 February 2023, an earthquake took place in the southern part of Turkey which was 398 times stronger than the Maheshkhali earthquake. What was the magnitude of the Turkey earthquake?

Polynomial Expression in Nature and Technology

You can learn from this experience-

- Construction process of polynomial expressions .
- Addition, subtraction, multiplication and division of polynomials
- Methods of factorization of polynomials.
- Factor theorem
- Factorization of perfect square expression
- Factorization of addition and subtraction of cubic expressions.
- Different methods convert to partial fractions.

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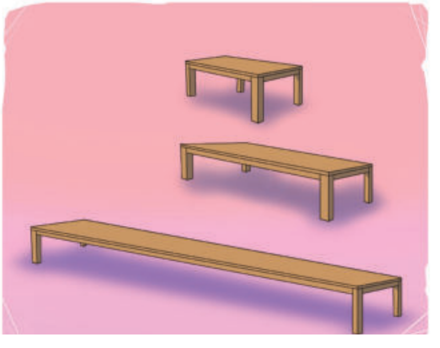
Polynomial Expression in Nature and Technology

Nature is full of mystery. Human beings have been trying to upgrade their wisdom by observing the creation of nature. Gradually, they have become scientists. Scientists invent and discover different things using their acquired knowledge according to their needs. Human beings have found that mountains were created for maintaining the equilibrium of earth. Based on this finding, the concept of sustainable architecture. In this chapter, we will try to find out where and how the model of polynomial expressions has been hidden behind the mighty nature and how these expressions have been used in technology.

Polynomial expressions are algebraic expressions. Problems related to numerical expressions are converted to algebraic expressions using variables for any number. After that the problem is solved through algebraic formulae and the result is used for any number. Let's learn how to form polynomial expressions from real life problems

1. Forming Polynomial Expressions from Real Life Problems

Minhaj's father ordered three tables from a carpenter. One was Minhaj's reading table; one was their dining table and the other one was for Minmhaj's younger sister's toys. The carpenter asked for the measurement of the tables. In that case, Minhaj's father asked for Minhaj's opinion. Minhaj is a student of class nine and he has some ideas about measurement. He told the carpenter that the length of her sister's table would be one unit less than twice the width. The length of his table would be one unit more than the square of the width and the length of their dining table would be one unit greater than the cube of the width minus twice the width. If the width of each table is x, the length of Minhaj's sister's table = $2x - 1$.



Individual task: Express the length of Minhaj's reading table and dining table through.

The equations obtained by using the variable x to measure the length above are polynomial expressions.

2. Polynomial Expression

In your previous class, you learnt about terms and variables of algebraic expressions. The variable of algebraic expression is a symbol that indicates any numeric expression. With the use of variables, we can convert the numeric expression into algebraic expression. When a variable refers to a particular number, it is called constant. The product of one or more variables and constant is a term of algebraic expression. The algebraic expression containing one or more terms is called a polynomial expression. In polynomial expressions, the summation of every term of a variable's index is considered the degree of the term. The term whose degree is zero is called constant term and the highest degree of those terms is called the degree of polynomial.

Example 1: $5x - 3$ is a binomial expression containing one variable. Here, -3 is a term whose degree is 0. It means -3 is a constant term. Again $5x$ is a term whose degree is 1 and 5 is the coefficient of x

Example 2: $xy - 5x + y$ is a trinomial expression containing two variables. Here, x and y are the two variables and there are three terms in this expression. xy is a term whose coefficient is 1.

Individual task:

Write down five polynomial expressions mentioning the number of variable and term. Find out the constant term and coefficient of each expression.

3. Polynomial Expression with one Variable

Here, we will discuss polynomial expression containing one variable (x). For example

1 3, $2x$, $-x^2$, x^4 etc. are the monomial expressions with the variable x.

2 $1 + 2x$, $-2 + x^4$ etc. are the binomial expressions with the variable x.

Now, we will discuss the general form of expressions. The general form of polynomial expression with one variable is:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0$$

It is indicated by $p(x)$). It means:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0$$

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Here,

- $a_n, a_{n-1}, \cdots, a_1, a_0$ are real numbers
- n is a non-negative (zero and positive) integer number. It is considered the degree of $p(x)$.
- *If* $n = 0$, $p(x) = a_0$. It is a constant quantity.
- $p(x) = 0$ is considered as a zero polynomial.
- In the polynomial expression p (x), $a_r x^r$ are the terms for any magnitude of n. It means that $a_1 x$ is a term; $a_2 x^2$ is a term. Here, a_0 is also a term that is constant in nature.
- a_n is a coefficient of x^n for every n. It means that a_1 coefficient of x; a_2 is a coefficient of x^2 .
- $a_n x^n$ is a prime term and a_n is a prime coefficient.

Individual task: **In the expression** $p(x) = 2x^2 - 3x + 1$, what is the degree, constant term, prime term, prime coefficient and the coefficient

The value of $p(x)$ that we get for any specific magnitude a of the variable x is referred by $p(a)$.

Individual task: If $p(x) = 5x^3 - 3x + 1$, find the value of $p(0)$, $p(1)$, $p(-1)$, $p(2)$ and $p(\frac{1}{2})$

Group work: Divide into four groups. Complete one task from the followings.

1. Write down linear polynomial expressions containing one variable with different terms. How many expressions did you write?
2. Write down quadratic polynomial expressions containing one variable with different terms. How many expressions did you write?
3. Write down cubic polynomial expressions containing one variable with different terms. How many expressions did you write?
4. Write down quartic polynomial expressions containing one variable with different terms. How many expressions did you write?

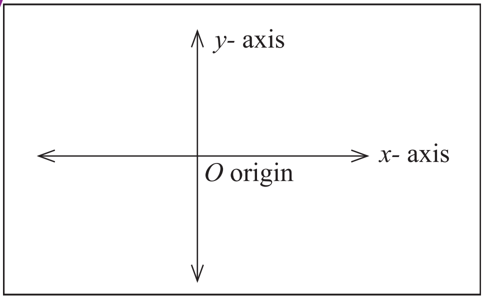
Have you found any relationship between the degree and number of terms of polynomial expression containing one variable after scrutinizing the expressions that you wrote in your previous task? If you have found any relationship, please write down about the relationship in the following box

4. The Graph of Polynomial Expression with one Variable

It becomes easier to observe a problem when a mathematical problem is expressed in geometric form. We can express polynomial expressions with one variable in geometric form through a graph. Different values of polynomial expressions are obtained for different values of variables and the values of polynomial expressions can be expressed by using two- dimensional coordinate geometry. The form expressed through two-dimensional coordinate geometry is called the graph of polynomials. So, before drawing the graph, we need to learn about two-dimensional coordinate geometry.

4.1 Two-dimensional Coordinate Geometry

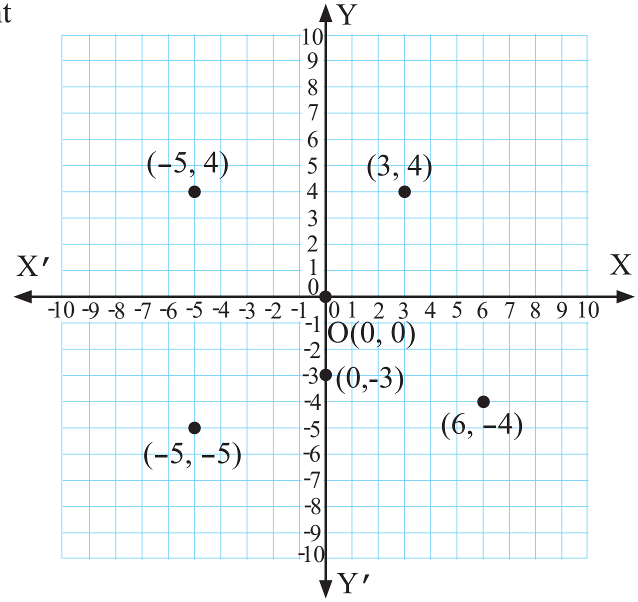
In two- dimensional coordinate geometry, one straight line is drawn horizontally and one straight line is drawn vertically. The horizontal line is called x-axis and the vertical line is called (x-axis). This plane is called xy-plane. The point where x-axis and y-axis intersect is called origin. The origin referred by O.



4.2 Position of a Point in xy-plane

In xy-plane, the position of a point is expressed by (a, b) where a is taken from x-axis and b is taken from y-axis. Here, a is called abscissa and b is called ordinate. Numbers to the right on the x-axis from the origin are positive and numbers to the left on the x-axis from the origin are negative. In a similar way, the numbers above on the y-axis from the origin are positive and the numbers below on the y-axis from the origin are negative. We can express any point of xy-plane through the number of x-axis and y-axis. In order to show the point (a,b) on xy-plane, we need to move a unit to the positive direction of the x-axis from the origin. After that we have to move we have to move b unit up to the positive direction of the y-axis from the origin. we got the point of (a,b) on xy-plane.

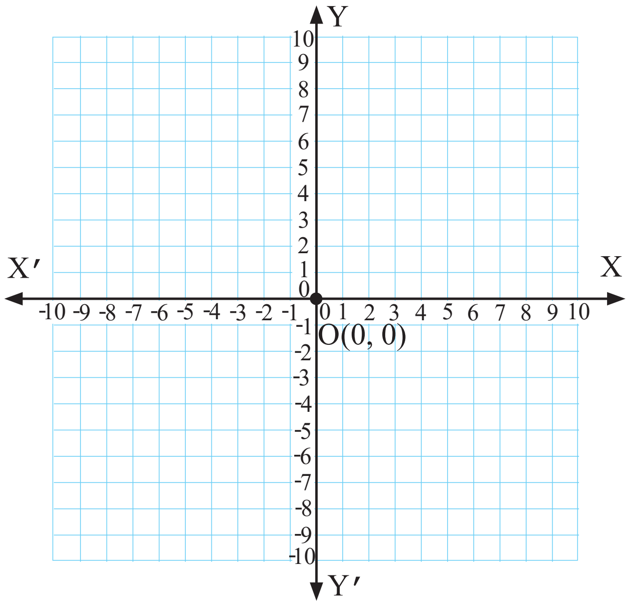
Example : In order to show the point positive direction of the x-axis from the origin. After that we have to move we have to move 4 unit up to the positive direction of the y-axis from the origin. The intersection we got is the point of (3,4) on xy-plane. The position of origin is expressed by (0,0). In this way, we can refer to any point on xy-plane. In the adjacent figure, the points (0, 0), (3, 4), (−5, 4), (−5, −5) and (6, -4) have been shown.



Pair work

Place the following points on the adjacent xy-plane. (3, 0), (1, 2), (0, 4), (−3, 5), (−6, 0),

(−4, −5), (0, −3), (4, −2)



4. 3 The Fundamental Way of Drawing the Graph of Polynomial Expressions with One Variable

Suppose, $p(x)$ is a polynomial expression. We have to find out the value of $p(x)$ for different values of x. If the value of x is a, the value of $p(x)$ will be $p(a)$ So, the point $(a,p(a))$ will be placed on the polynomial expression $p(x)$ on xy -plane. After finding out the value of $p(x)$ for different values of x and placing the points considering the values of x and $p(x)$, we have to add the points through a straight line. This will be the graph of polynomial expression $p(x)$.

To draw a graph of a polynomial expression is quite difficult. In many cases, it is nearly impossible. In the next grade, you will learn how to draw different graphs. We are lucky that we are living in an era of digital technologies. We can draw a graph of polynomial expression by using graphics calculation, computers, even by using a smartphone. Do you know how these devices draw a graph? The technologists have set a program regarding the fundamental way of drawing graphs that is able to make a graph by placing several points within a moment. When you will get older, you will also contribute to ICT that will make people's life easier. Now we will discuss the way of drawing the graph of small polynomial expressions containing power. In polynomial expressions, we will use *a, b, c* as the coefficients.

4.4 The Graph of Linear Polynomial Expression

The form of linear polynomial expression is:

$$p(x) = ax + b, \qquad a \neq 0$$

It is easy to draw the graph of linear polynomial expression because it indicates a straight line. Finding out two different point is enough to draw the graph of a linear polynomial expression $p(x)$. In order to draw the graph of linear polynomial expression$p(x)$, we have to find out two values of $p(x)$ for two different values of x After that we have to place two points on xy -plane considering the value of x and $p(x)$ Finally, a straight line has to be drawn for connecting the two points. This straight line would be the graph of the linear polynomial expression $p(x)$.

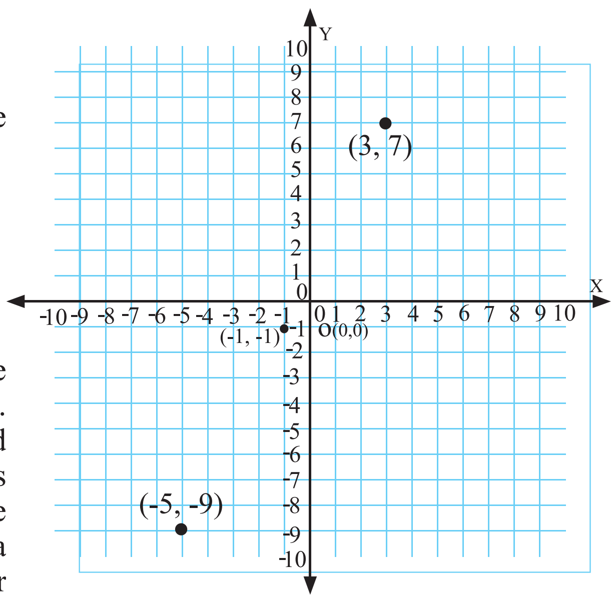
Example:

Draw a graph for the linear polynomial expression $p(x) = 2x + 1$

Solution: Suppose, $y = p(x) = 2x + 1$.

Find out two values of y in reference to two values of x and complete the table below

x	-3	0
y	-5	1
(x, y)	$(-3, -5)$	$(0, 1)$



Place the points obtained from the above table on the graph paper given above. Three points have already been placed for your convenience. Any two points would be enough here. Now, connect the points. What can you see? You will see a straight line. So, it is clear that the linear polynomial expression $2x + 1$ refers to a straight line.

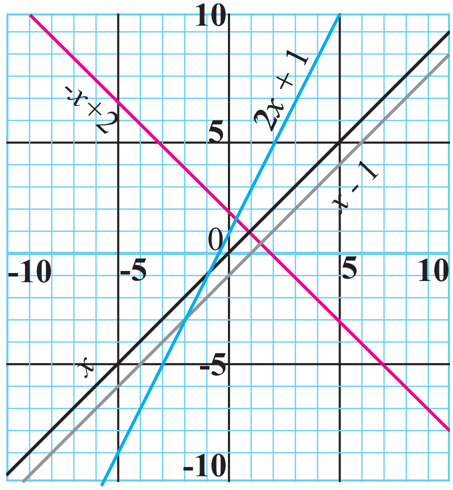
Pair work:

Create graphs for the followinglinear

polynomial expressions.

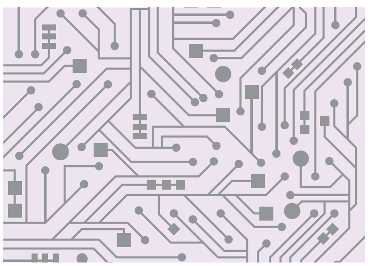
1) $x - 1$, 2) x, 3) $-x + 2$

Match your graphs with the adjacent figure. In this case, you may use your digital device too. If your graphs do not match, you have to find out the mistake and make corrections.



4.5 Linear Polynomial Expressions in Nature and Technology

The geometric forms of linear polynomial expressions are similar to different objects of nature. The leaves of different plants look linear, such as coconut leaves, palm leaves, betel leaves etc. These leaves are neatly organized. No line intersects another line. We need to learn about the linear polynomial expression if we want to understand the characteristics of natural linear objects.



Several household objects are made in a linear manner, for example, table, chair. window, doors are made of linear wood. Technology is highly influenced by straight lines. The circuit design of different digital devices is linear. In order to create the mathematical model of these objects, we need the knowledge of linear polynomial expression.

4.6 The Graph of Quadratic Polynomial Expressions

The general form of quadratic polynomial expression is:

$$p(x) = ax^2 + bx + c, \qquad a \neq 0$$

Drawing the graph of quadratic polynomial expression is not as easy as linear polynomial expression because it doesn't indicate a straight line. In order to draw the graph of quadratic polynomial expression $p(x)$, we need to find out several points. In case of taking the values of x, we have to understand for which two values of x the value of

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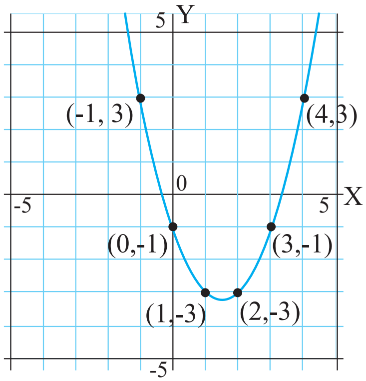
$p(x)$ is equal. If the value of $p(x)$ is 0 for any value of x, we have to consider all these values of x. We have to find out the value of $p(x)$ in reference to the different values of x. After placing the points considering the value of x and $p(x)$, the points have to be connected through a line. This line will be considered the graph of the quadratic polynomial expression $p(x)$



Example: Draw a graph for the quadratic polynomial expression. $p(x) = x^2 - 3x - 1$

Solution: Suppose, $y = p(x) = x^2 - 3x - 1$. Now find out the value of y in reference to different values of x and identify the points of (x,y). Complete the following table.

x	-1	0	1	2	3	4
y	3	-1	-3	-3	-1	3
(x, y)	$(-1, 3)$	$(0, -1)$	$(1,-3)$	$(2, -3)$	$(3, -1)$	$(4, 3)$

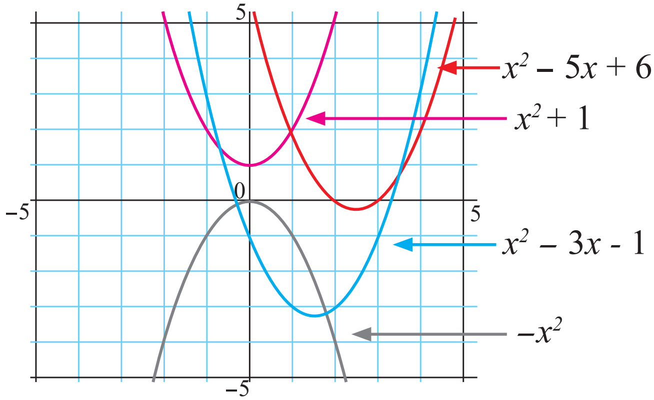
Now place the points obtained from the above table on the adjacent graph paper. Connect the points with a curve. Here, the values of x are 1 and 2 and $y = -3$. As a result, the curve will touch the value $\frac{1+2}{2} = 1.5$ of x and turn. We will get a curve just like the above graph. This curve indicates the quadratic polynomial expressionc $p(x) = x^2 - 3x - 1$

Pair work

Draw graphs for the following quadratic polynomial expressions.

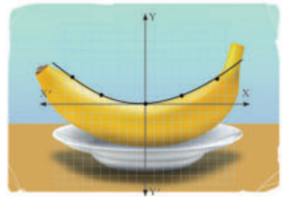
1) $x^2 - 5x + 6$, 2) $-x^2$, 3) $x^2 + 1$

These expressions are quadratic polynomials. Their geometric forms are just like the curves drawn on the adjacent graph paper. Match your graphs with the adjacent figure. In this case, you may use your digital device too. If your graphs do not match, you have to find out the mistake and make corrections.



4.7 Quadratic Polynomial Expressions in Nature and Technology

Look at the form of mountains and bananas. Their forms are similar to quadratic polynomial expressions. Their forms can be expressed through quadratic polynomial expressions. We need to learn about the quadratic polynomial expression if we want to understand the characteristics of natural objects.



We can see many man made things that have the influence of quadratic polynomial expressions, such as bridge, gate etc. This kind of strong construction is made based on these mathematical models. In order to create such mathematical models, the knowledge about quadratic polynomial expression is needed.

Individual task:

Give five examples of quadratic polynomial expressions. Present the geometric form of your examples and explain exactly where these forms can be found in nature and technology.

4.8 The Graph of Cubic Polynomial Expressions

The general form of cubic polynomial expression is:

$$p(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

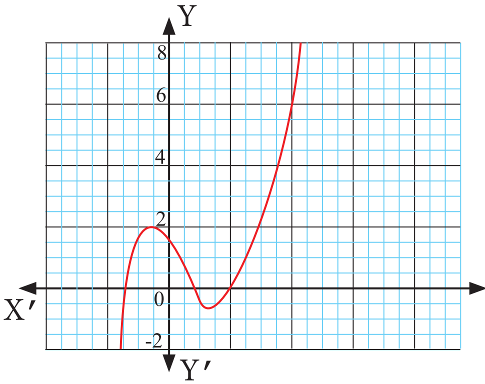
It is very difficult to draw the graph of cubic polynomial expressions. it does not indicate any straight line. For this reason, we need to learn about the characteristics of cubic polynomial expressions. Here, we will take several values of x for drawing the graph of $p(x)$. We will find the value of $p(x)$ considering the values of x. Then we will find the points $(x, p(x))$. If the value of $p(x)$ is 0 for any value of x, we have to consider all these values of x. After that we have to place the points on xy-plane and connect the points with a line. This line will be the graph of cubic polynomial expression $p(x)$.

Example:

Draw a graph for the cubic polynomial expression $p(x) = x^3 - 2x^2 + 2x - 1$

Solution: Suppose, $y = p(x) = x^3 - 2x^2 - x + 2$. Now find out the value of y in reference to different values of x and identify the points of (x,y). Complete the following table.

x	-1	0	1	2	3
y	0	2	0	0	8
(x, y)	$(-1, 0)$	$(0, 2)$	$(1, 0)$	$(2, 0)$	$(3, 8)$



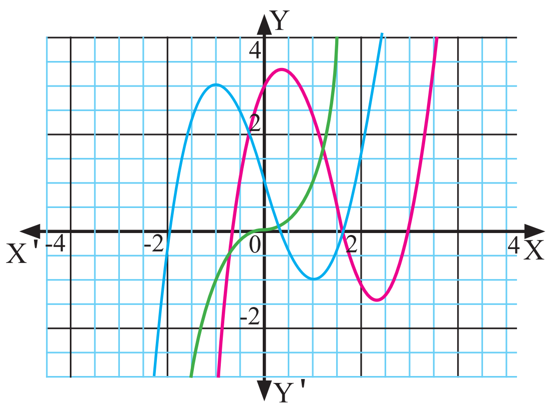
Now place the points obtained from the above table on the adjacent graph paper. Connect the points with a curve. This curve is the graph of the cubic polynomial expressions $p(x) = x^3 - 2x^2 - x + 2$

Pair work

Draw the graph of the following expressions.

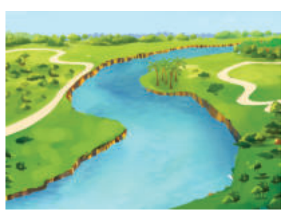
1) $x^3 - 4x^2 + 2x + 3$, 2) $x^3 - 3x + 1$, 3) x^3

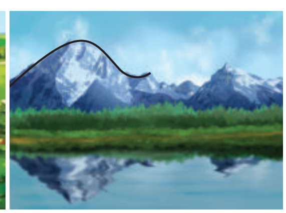
These expressions are cubic polynomials. Their geometric forms are just like the curves drawn on the adjacent graph paper. Match your graphs with the adjacent figure. In this case, you may use your digital device too. If your graphs do not match, you have to find out the mistake and make corrections



4.9 Quadratic Polynomial Expressions in Nature and Technology

Think of the course of the rivers or the height of the mountain peaks. Their forms are similar to cubic polynomial expressions. These natural structures can be expressed through cubic polynomial expressions. We need to learn about the cubic polynomial expression if we want





to understand the characteristics of natural c

We can see many man made things that have the influence of cubic polynomial expressions, such as large bridges, gates etc. This kind of strong construction is made based on these mathematical models. In order to create such mathematical models, the knowledge about cubic polynomial expression is needed.

Individual task:

Give three examples of quadratic polynomial expressions. Present the geometric form of your examples and explain exactly where these forms can be found in nature and technology.

5. Polynomial Expression with Two Variables

Real life problem 1

In the market, rice and pulses of different prices are available. If the price of 1 kg rice is x Taka and the price of 1 kg pulses is y Taka, what is the price of 6 kg rice and 2 kg pulses?

$$\text{Price} = 6x+2y \text{ Taka}$$

This a polynomial expression with two variable where the variables are x and y.

Real life problem 2

If the length of a land is x and the width is y, what is the area of that land?

$$\text{Area} = xy$$

This a polynomial expression with two variable where the variables are x and y.

In this way, different real life problems can be expressed through the polynomial expression with two variables. Some examples are given below.

1. $x - 3y + 6$ 2. $xy - 1$ 3. $x^2 + y^2 - xy$ 4. $x^3 - x^2y^2 + x - y + 5$

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5.1 General Form of Polynomial Expression with Two Variables

Expression containing x and y variables is referred by $p(x, y)$. Generally, the form of polynomial expression with two variable is $ax^m\ y^n$ Here:

- m and n are non-negative and integer numbers.
- $x^m y^n$ is the coefficient of a.
- If $m = 0, n = 0$, then $ax^m y^n = a$ is a constant.
- $m + n$ is the degree of $ax^m y^n$ and the degree of the constant term is 0.

The highest degree of the terms of polynomial expression $p(x, y)$ is called the degree of $p(x, y)$.

Example: Find out the coefficient and degree of every term of the polynomial expression $p(x, y) = x^3 - x^2y^2 + 5x$

Solutions: The coefficient of x^3 is 1 and the degree is 3.

- The coefficient of $-x^2y^2$ is -1 and the degree is $2 + 2 = 4$
- The coefficient of $5x$ is 5 and the degree is 1.

So, the degree of $p(x, y) = x^3 - x^2y^2 + 5x$ is 4.

Pair work: Find out the coefficient and degree of every term of the following polynomial expressions. What is the degree of the expression?

1. $x^4 - 5x^2y^2 + 3x$
2. $x^2y^2 - 5xy^3 + y^4$
3. $xy + 3y - 5$
4. $x^2 + 2xy - 3y^2 + 5x - 2y + 3$

6. Polynomials of three variables

Polynomial expressions with three variables are formed from real life problems just like the polynomial expressions with two variables.

Real life problem: What is the sum of the volumes of three cubes whose lengths are x, y and z?

Solution: We know that x^3 is the volume of the cube whose length is x.

The sum of the volume of three cubes containing the length x, y and z is $x^3 + y^3 + z^3$

This is a polynomial expression with three variables.

6.1 General Form of Polynomial Expression with Three Variables

Expression containing x, y and z variables is referred by $p(x, y, z)$. Generally, the form of polynomial expression with three variable is $ax^m y^n z^p$ The degree of general term is $m + n + p$ and the highest degree of the terms of polynomial expression $p(x, y, z)$ is called the degree of $p(x, y, z)$.

Example: Find out the coefficient and degree of every term of the polynomial expression $p(x, y, z) = x^3z - x^2y^2 + 2xz^3$. What is the degree of the expression?

Solution:

- The coefficient of $x^3 z$ is 1 and the degree is 3+1=4
- The coefficient of $-x^2y^2$ is-1 and the degree is 2+2=4
- The coefficient of $2xz^3$ is 2 and the degree is 1+3=4

So, the degree of the expression $p(x, y, z) = x^3 z - x^2y^2 + 2xz^3$ is 4.

Pair work: Find out the coefficient and degree of every term of the following polynomial expressions. What is the degree of the expression?

1. $x^3 - 10xy^2 z + 2x^2 z + 1$
2. $x^2y^3 z - 7x^3y^3 + 3y^4z$
3. $5xyz + 2xy^2 - 5y + 3z$
4. x2y2 z + 2yz3 - 3y2 + 5xy - 2z + 2

7. Polynomials with Special Characteristics

You have noticed that there are many polynomial expressions. Many of these polynomials have complicated characteristics. If we know about the characteristics of polynomial expressions, it would be easier for us to use them for different purposes. Now, we will discuss some characteristics of polynomial expressions.

7.1 Homogeneous Polynomial

In different polynomial expressions, you have noticed that there are many polynomials whose degree of every term is equal. This kind of polynomial is called homogeneous polynomial, for example:

1. $x + y$ is a homogeneous polynomial with two variables whose degree of every term is 1.
2. $x^2 + 2xy + y^2$ is a homogeneous polynomial with two variables whose degree of every term is 2.

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3. $x^2 - 3xz + 2yz - xy + y^2$ is a homogeneous polynomial with three variables whose degree of every term is 3.

Pair work:

1. Give 5 examples of homogeneous polynomials of two variables containing different term numbers whose degree of every term is 2
2. Give 5 examples of homogeneous polynomials of two variables containing different term numbers whose degree of every term is 3.
3. Give 5 examples of homogeneous polynomials of three variables containing different term numbers whose degree of every term is 2.
4. Give 5 examples of homogeneous polynomials of three variables containing different term numbers whose degree of every term is 3.

7.2 Symmetric Polynomial

When two variables interchange their position and the expression remain unchanged, the polynomial is called symmetric polynomial, for example:

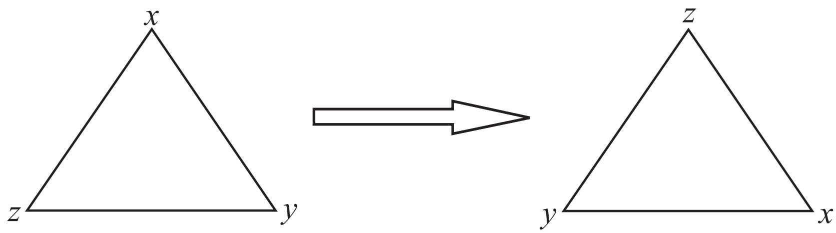
1. $x + y$ 2. xy 3. $x^2 + y^2 - x - y + 1$ 4. $xy + yz + zx$

Pair work:

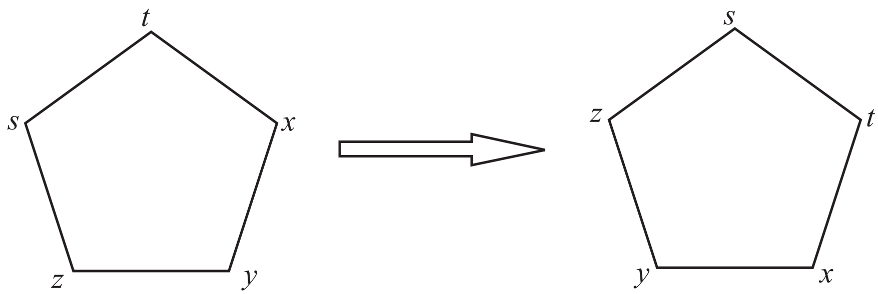
1. Why are the above expressions considered symmetric polynomials? Explain.
2. Give 5 examples of homogeneous polynomials who are not symmetric.

7.3 Cyclic Polynomial

Let's think of an expression $x + y + z + xyz$. If y is replaced with x, z is replaced with y and x is replaced with z and the expression remains unchanged, we can call it a cyclic polynomial. When three or more than three variables interchange their position and the expression remains unchanged, the polynomial is called cyclic polynomial. We can show the interchange of position through geometry. For example, the cyclic displacement of the variable x, y, z are as follows:



The cyclic displacement of the variables x, y, z, s, t is as follows.



Example:

1. $x^2 + y^2 + z^2$ is a cyclic polynomial expression with three variables. Here, the degree of every term is 2. So, the degree of the expressions is 2.

2. $x^3 + y^3 + z^3 + w^3 - 3xyzw$ is a cyclic polynomial expression with four variables. Here the highest degree of a term is 4. So, the degree of the expression is 4.

Pair work:

1. Give an example of a simple cyclic polynomial expression with three variables.

2. Give an example of two dimensional polynomial expressions with four variables.

8. Addition, Subtraction, Multiplication and Division of Polynomials

The variable of the polynomial indicates the numeric expression. Therefore, we can add, subtract, multiply and divide polynomials like the numeric expression.

8.1 Addition and Subtraction

In case of addition or subtraction of two polynomials with one variable, the coefficients of the terms that have the same degree have to be added or subtracted.

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Example: If $p(x) = x^3 - 3x + 1$ and $q(x) = 2x^3 - x^2 + 3$, then what is result of $p(x) + q(x)$ and $p(x) - q(x)$?

Solution: Complete the following table.

Expression	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant `term
$p(x)$	1	0		
$q(x)$				3
Sum of coefficient	3			
$p(x) + q(x)$	$= 3x^3 - x^2 - 3x + 4$			
$p(x) - q(x)$				

8.2 Multiplication

If *a,b,c* are the real numbers, then

$$a(b+c)=ab+ac$$

This is called distributive law.of real numbers

With the help of this law, we can write that if a,b,c,d are the real numbers-

$(a+b)(c+d)=ac+ad+bc+bd$

As the variables in polynomials are the real numbers, we can use this law/rule for polynomials. We can multiply or divide the polynomials like the real numbers. Previously, you learnt how to multiply polynomials. You learnt the multiplication of polynomials with the degree 0 and 1. We can multiply any polynomial by using the law of variables and the above formula.

Example: Multiply x^2+3 by $x+2$.

Solution : (x2 + 3)(x + 2) = x2.x + 2x2 + 3x + 6

= x3 + 2x2 + 3x + 6

Individual task

1. Multiply $2x + 3y$ by $3x + 2y$.
2. Multiply $x^2 y + 5y - 1$ by $x^2 + y^2$

8.3 Division

You learnt how to do long division of numeric expressions. The process of diving 12 by 5 is as follows:

$$\begin{array}{l} 5)\ 12\ (2 \\ \quad \underline{10} \\ \quad\ \ 2 \end{array}$$

Here, the dividend is 12, divisor is 5, 2 is the quotient and the remainder is 2. Just like numeric expressions, we can go for a long division.

Example 1 $x - 1)\ 4x^2 - 4 \quad (4x + 4$

$$\begin{array}{rl} & 4x^2 - 4x \\ \hline (-) & \\ & 4x - 4 \\ & 4x - 4 \\ \hline (-) & 0 \end{array}$$

Example 1 $2x^2 - 1)\ 4x^2 + 1 \quad (2$

$$\begin{array}{rl} & 4x^2 - 2 \\ \hline (-) & 3 \end{array}$$

Individual task

1. Divide $x^4 - 3x^2 + 5$ by $x^2 - 2$
2. Divide $x^3 + 5x - 6$ by $x - 1$

8.4 General Characteristics of the Process of Division

If *p(x)* and *d(x)* are two polynomials [where $d(x) \neq 0$], then

$$\frac{p(x)}{d(x)} = q(x) + \frac{R(x)}{d(x)}$$

where $q(x)$ and $R(x)$ are polynomials and $p(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient and $R(x)$ is the remainder.

- The degree of $R(x)$ is less than the degree of $q(x)$.
- If the degree of $d(x)$ is greater than the degree of $p(x)$, then $q(x) = 0$.

By multiplying $d(x)$ in the both sides of the above equation, we get:

$$p(x) = d(x)\, q(x) + R(x) \; \ldots \ldots \ldots \; (1)$$

It means that dividend = divisor × quotient + remainder

Individual task

1. If we divide $x^3 - x^2 + 2$ by $x^2 - 2$, what will be the remainder?
2. If we divide $x^5 + 5x^3 - 6x - 2$ by $x^3 - x + 1$ what will be the remainder?

9. Remainder Theorem

If we want to get the geometric form of a polynomial expression with one variable, we have to find out the value of that expression in reference to different values of variables. In this case, we can easily get the value through the remainder theorem. Let's see the process through an example.

Suppose, $p(x) = x^4 - 3x^2 + 2x - 2$ and we want to get the value of $p(x)$ in reference to different values of x.

- If the value of x is 0, what will be the value of $p(x)$? The value will be -2. It means $p(0)$=-2. Again divide $p(x)$ by x. The remainder will be -2. It means the remainder is $p(0)$
- If the value of x is 1, what will be the value of $p(x)$? The value will be -2. It means $p(1)$=-2. Again divide $p(x)$ by x-1. The remainder will be -2. It means the remainder is $p(1)$
- If the value of x is-2, what will be the value of $p(x)$? Find out the value. Again divide $p(x)$ by x+1. The remainder will be $p(-1)$.

Do you find any relationship between divisor and quotient after analyzing the above results? The relationship that you get is regarded as the remainder theorem.

Remainder theorem: If positive polynomial expression $p(x)$ is divided by $(x-a)$, the remainder will be $p(a)$.

We can prove the remainder theorem very easily.

Proof: What we get from the above relationship is that

$$p(x) = d(x)q(x) + R(x)$$

If the divisor is $q(x)=x-a$, then

$$p(x)=d(x)(x-a)+R(x)$$

As the degree of $q(x)$ is 1, $R(x)$ is a constant. Suppose, $R(x)=R$. Therefore,

$$p(x)=d(x)(x-a)+R$$

Now, if, $x=a,$

$$p(a)=d(a)(a-a)+R=R$$

It means that $p(a)$ is equal to the remainder R.

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Note that we can find out the remainder without dividing a polynomial expression by a simple expression (*ax+b* where *a≠0)*. In this case, if we divide the positive polynomial expression *p(x)* by *ax+b*, the remainder will be $p\left(-\frac{b}{a}\right)$

Example: If the polynomial expression $3x^3 - 2x + 1$ is divided by $2x + 1$, what will be the remainder? Find out the remainder by using the remainder theorem.

Solution: According to the remainder theorem, the remainder will be:

$$p\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 1 = -\frac{3}{8} + 2 = \frac{13}{8}.$$

Individual task:

1. If $x^2 - 4x + 3$ is divided by $x - 3$, what will be the remainder? Find out the remainder by using the remainder theorem.
2. If $2x^4 - x^2 + 2$ is divided by $3x - 2$, what will be the remainder? Find out the remainder by using the remainder theorem.

10. Factorization

Factor plays a vital role in solving real life problems. We can find out the value of variables through factors. That is why factor is a very important issue in algebra. When the factor m is a linear expression, it is very easy for us to find out the real value of the variable/s. So, the factorization should be in linear expression. In the previous class, you learnt about the factorization of algebraic expression. Now, we would like to discuss the process of factorization of polynomials.

If a polynomial expression can be written as the product of several polynomials, the multiplied polynomials are considered the factor of that polynomial, for example $x^2 + 1$ is a factor of $x^4 - 1$ Do you know why $x^2 + 1$ is a factor of $x^4 - 1$ If $x^4 - 1$ is divided by $x^2 + 1$, the remainder will be 0. This the reason why $x^2 + 1$ is a factor of $x^4 - 1$ What other factors can $x^4 - 1$ have? Write down your answer in the following table.

If we express as a product of prime polynomials of a polynomial is called factorization of that polynomial.

Example: What we get from the factorization of $x^4 - 1$ is that

$$x^4 - 1 = (x^2 + 1)(x + 1)(x - 1)$$

Here, we have got two linear expressions $x+1$ and x-1 as the factors. A quadratic expression $x^2 + 1$ is also a factor. This quadratic expression cannot be factorized in real linear expression. In factorization, it is very important to express the factors as the product of linear expression.

10.1 Factor Theorem

x-a will be the factor of $p(x)$ if only $p(a)$=0.

Proof:

Suppose, $p(x)$ is a polynomial with one variable. If x-a is a factor of $p(x)$, after dividing $p(x)$ by (x-a), the remainder will be 0. But according to remainder theorem, if we divide $p(x)$ by (x-a), the remainder will be $p(a)$. It means that $p(a)$=0. On the other hand, if $p(a)$=0, after dividing $p(x)$ by (x-a), we will get 0 as the remainder. It means that (x-a) will be a factor of $p(x)$.

Individual task:

Find out the values of x for what the values of the following polynomials are 0 and find out the factors. Go for factorization.

1. $x^2 - 5x - 14$
2. $3x^2 + 4x - 4$

10.2 Common Factor

If there is a common factor in every term of a polynomial, the factor should be extracted first for the sake of easy factorization. For example:

$$y^2 + xy + 3y = y(y + x + 3)$$

Here, y is the common factor in every term of $y^2 + xy + 3y$ Note that the other factor $(y + x + 3)$ is a prime factor. Therefore, it is a factorization.

Example: Factor the expression $x^2 + 3y^3 - xy - 3xy^2$

Solution: $\mathbf{x^2 + 3y^3 - xy - 3xy^2}$

$=x^2 - xy - 3xy^2 + 3y^3$

$= x(x - y) - 3y^2 (x - y)$

$= (x - y)(\mathrm{x} - 3y^2)$

10.3 Factorization of Perfect Square Expression

There are some polynomials that can be expressed in the form of a perfect square. If you notice carefully, you will understand which expressions should be expressed in perfect square. For example:

$$x^2 + 6xy + 9y^2$$

Do you think it can be expressed in the form of a perfect square? It can be written:

$$x^2 + 2.x.3y + (3y)^2 = (x + 3y)^2$$

This kind of expression can be expressed as the product of the same expression's factors. It means that

$$(x + 3y)^2 = (x + 3y)(x + 3y)$$

Individual task

Factor the expression in perfect square.

$$x^2 + y^2 + 2xy + 2x + 2y + 1$$

10.4 Factorization of the Expression Expressed in the Difference of Two Squares

We can convert a polynomial into the difference of two expressions by using the following formula.

$$x^2 - y^2 = (x + y)\,(x - \mathrm{y})$$

Example: Factor the expression $x^2 + 4x + 1$

Solution: $x^2 + 4x + 1 = x^2 + 2.x.2 + 2^2 - 3 = (x + 2)^2 - (\sqrt{3}\,)^2$

According to the above formula, we can write that,

$$x^2 + 4x + 1 = (x + 2 + \sqrt{3}\,)\,(x + 2 - \sqrt{3}\,)$$

Here, $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are the irrational numbers.

Individual task: Factor the expression $a^4 + 4b^4$

10.5 Factorization of quadratic expressions

The polynomial like $ax^2 + bx + \mathrm{c}$, [$\mathrm{a} \neq 0$] Now, we will discuss how to factor this kind of expression. play an important role in creating and solving real life problems.

10.5.1 Factorization through Splitting of the Middle Term

We can factor the expression $x^2 + ax + b$ by splitting its coefficient a in a specific way. In this case, we have to divide the value of a into c and d in such a way where their sum is equal to a and their product is equal to b. Then we can go for the factorization by using the following formula.

$$\boxed{x^2 + (c + d)\,x + cd = (x + c)(x + d)}$$

If the expression is $ax^2 + bx + c$, the value of b has to be divided into d and e in such a way where the their sum is equal to b and their product is equal to ac. After that we can go for the factorization by finding out the common factors. This is the process of factorization through splitting the middle term.

Example: Factor the quadratic polynomial $x^2 + 3x + 2$ by splitting the middle term.

Solution: If we compare $x^2 + 3x + 2$ with $ax^2 + bx + c$, we get $a = 1$, $b = 3$, $c = 2$

Can you divide b = 3 into two different numbers where the sum of the two numbers is 3 and their product is equal to $a.c = 1 \times 2 = 2$?

The numbers are 2 and 1. So, we can write:

$x^2 + 3x + 2 = x^2 + (2 + 1)\,x + 2 \times 1$

$= (x + 2)\,(x + 1)$ [according to formula]

Example: Factor the quadratic polynomial $2x^2 + 3x - 2$ by splitting the middle term.

Solution: If we compare $2x^2 + 3x - 2$ with $ax^2 + bx + c$, we get $a = 2$, $b = 3$, $c = -2$

Can you divide $b = 3$ into two different numbers where the sum of the two numbers is 3 and their product is equal to $a.c = 2 \times (-2) = -4$?

The numbers are 4 and -1. So, we can write

$$2x^2 + 3x - 2 = 2x^2 + 4\text{x} - x - 2$$
$$= 2x(x + 2) - 1(x + 2)$$
$$= (2x - 1)(x + 2)$$

Pair work: Factor the following expressions.

1. $x^2 - 5x + 6$ 2. $3x2 + 5x + 2$

10.5.2 Factorization in a General Way

Sometimes, we cannot split the middle term of the polynomial like $ax^2 + bx + c$, $[a \neq 0]$ according to our convenience.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left\{x^2 + 2.\text{x}.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right\}$$

$$= a\left\{\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\right\}$$

$$= a\left\{\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\right\}$$

Here the polynomial expression has been factored through linear polynomial expression. If the value of $b^2 - 4ac$ is negative, it means:

$b^2 - 4ac < 0$, the quadratic polynomial cannot be factored through linear polynomial expression. In this case, the quadratic polynomial expression $ax^2 + bx + c$, $[a \neq 0]$ is a prime expression that means $ax^2 + bx + \mathrm{c}$, $[a \neq 0]$ cannot be factored in reference to the real value.

Pair work: Examine whether the following expressions can be factored. Factor the expressions that are doable.

1. $x^2 + 1$ 2. $x^2 - 10x + 25$ 3. $x^2 - x + 5$ 4. $3x^2 - 7x + 3$

10.6 Factorization of the Sum of Two Cubic Expressions

We know that

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

By using this formula, we can factor a cubic polynomial expression.

Example: Factor $x^3 + 8y^3$.

Solution: $x^3 + 8y^3 = x^3 + (2y)^3$

$= (x + 2y)(x^2 - x.2y + (2y)^2)$

$= (x + 2y)(x^2 - 2xy + 4y^2)$

Individual task:

Factor the following expression by using the formula of the sum of cubic expressions

1. $8x^3 + 27y^3$ 2. $x^3 + \frac{1}{x^3}$ 3. $x^3 + 3x^2 + 3x + 9$

10.7 Factorization of the Subtraction of Two Cubic Expressions

We know that

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

By using this formula, we can factor a cubic polynomial expression.

Example: Factor $x^3 - 64y^3$.

Solution: Factor $x^3 - 64y^3 = x^3 - (4y)^3$

$$= (x - 4y)(x^2 + x.4y + (4y)^2)$$

$$= (x - 4y)(x^2 + 4xy + 16y^2)$$

Individual task: Factor the following expression by using the formula of the subtraction of cubic expressions.

1. $27x^3 - 125y^3$ 2. $8x^3 - \frac{1}{8x^3}$ 3. $x^3 - 6x^2 + 12x - 7$

Identity

Let, $p(x)$ and q(x) be two polynomial expressions. If for all values of x, $p(x) = q(x)$ is satisfied; then it is called identity. This is expressed by $p(x) \equiv q(x)$. The sign '≡' is called identity sign

Example- Let $p(x) = (x + 1)^2$ and $q(x) = x^2 + 2x + 1$. Then,$p(x) = q(x)$, that is ,$(x + 1)^2 = x^2 + 2x + 1$ is an identity. Because, this equation is satisfied for all x

Note that, $x^2 + 2x + 1 = 0$ is not an identity. Because this equation is not true for all values of x. For example, if x=1, the left side =4 and the right side =0.

10.8 Partial Fractions

The expression $\frac{p(x)}{q(x)}$ (where $p(x)$ and $q(x) \neq 0$ are polynomials) is a rational fraction. Sometimes rational fractions are divided into several fractions for convenience. These divided multiple fractions are called partial fractions.

Example:

Rational fraction $\frac{3x + 1}{x^2 - 1}$ has to be divided into two parts where the denominator is a linear polynomial expression. Look at the fraction

$$\frac{3x + 1}{x^2 - 1} = \frac{3x + 1}{(x - 1)(x + 1)} = \frac{1}{x + 1} + \frac{2}{x - 1}$$

Here, $\frac{1}{x + 1}$ and $\frac{2}{x - 1}$, are the partial fractions of $\frac{3x + 1}{x^2 - 1}$

10.8.1 Different Methods to convert Partial Fraction

Rational fraction $\frac{p(x)}{q(x)}$ can be expressed in partial fractions in many ways. It depends on the degree of $p(x)$ and $q(x)$.

Methods of converting proper fractions

When the degree of $p(x)$ is less that the degree of $q(x)$, there will be no common factor. In this case, $\frac{p(x)}{q(x)}$ is considered a proper fraction, for example $\frac{3x+1}{x^2-1}$ is a proper fraction. This kind of expression can be expressed in partial fractions in the following way.

1. When $q(x)$ has only linear factors and the factors are not repeated, partial fraction for every factor of $q(x)$ has to be found and constant has to be set as the numerator of every partial fractions.

Example: Express $\frac{3x+1}{x^2-1}$ in partial fraction.

Solution: Here, $p(x) = 3x + 1$ and $q(x) = x^2 - 1$. If we factor $q(x)$, we get $q(x) = (x - 1)(x + 1)$ Note that $q(x)$ has only two linear factors and the factors are not repeated.

Therefore, after factoring $q(x)$ we get: $\frac{3x+1}{x^2-1} \equiv \frac{3x+1}{(x-1)(x+1)}$

Suppose,

$$\frac{3x+1}{(x-1)(x+1)} \equiv \frac{A}{x+1} + \frac{B}{x-1}$$

After multiplying $(x - 1)(x + 1)$ in the both sides, we get:

$$3x + 1 = A(x - 1) + B(x + 1).$$

If $x = -1$, $-2 = -2A$. It means that $A = 1$.

If $x = 1$, $4 = 2B$. It means that $B = 2$.

Therefore,

$$\frac{3x+1}{x^2-1} \equiv \frac{3x+1}{(x-1)(x+1)} \equiv \frac{1}{x+1} + \frac{2}{x-1}$$

Notice it carefully. The result is similar to the partial fraction of $\frac{3x+1}{x^2-1}$ that you found out earlier.

2. When q(x) has only linear factors and the factors are repeated, we get partial fractions from every repeated factor. After that we have to set constant in for the numerator of every partial fraction.

Example: Express $\frac{x+2}{(x-1)(x^2-1)}$ in partial fraction.

Solution: Here, $q(x) = (x-1)(x^2-1) = (x-1)(x-1)\,(x+1) = (x-1)^2\,(x+1)$.It means that the factor x-1 has two repetitions.

Suppose,

$$\frac{x+2}{(x-1)(x^2-1)} \equiv \frac{x+2}{(x-1)^2\,(x+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

After multiplying $(x-1)^2\,(x+1)$ in the both sides, we get:

$$x+2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Now, the value of x is -1. Find out the value A, B, C by setting 1 and 0.

We will get: $A = -\frac{1}{4}$, $B = \frac{3}{2}$ and $C = \frac{1}{4}$ So,

$$\frac{x+2}{(x-1)(x^2-1)} \equiv \frac{1}{4(x-1)} + \frac{3}{2(x-1)^2} + \frac{1}{4(x+1)}$$

3.When $q(x)$ has linear and quadratic factors and the factors are not repeated, we have to set linear polynomial expression as the numerator of the quadratic factor $q(x)$.

Example: Express $\frac{x+2}{(x-1)(x^2+2)}$ as partial fraction.

Solution: Suppose,

$$\frac{x+2}{(x-1)(x^2+2)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

After multiplying $(x-1)(x^2+2)$ in the both sides, we get:

$$x + 2 \equiv \mathrm{A}(x^2+2) + (\mathrm{B}x + C)(x-1)$$

Now think.

Find out the value of A, B, C. The values are: A = 1, B = −1 and C = 0.

So,

$$\frac{x+2}{(x-1)(x^2+2)} \equiv \frac{1}{x-1} - \frac{x}{x^2+2}$$

4. When q(x) has quadratic factors and the factors are repeated, we get partial fractions from every repeated quadratic factor. We have to set linear polynomial expression as the numerator of every partial fraction.

Example: Express $\dfrac{x+1}{(x-1)(x^2+1)^2}$ in partial fraction.

Solution: Suppose,

$$\frac{x+1}{(x-1)(x^2+1)^2} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

After multiplying $(x-1)(x^2+1)^2$ in the both sides, we get:

$$x + 1 \equiv \mathrm{A}(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)$$

Now think. And find out the value of A, B, C The values are:

$A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, D=-1 and $E = 0$.

So,

$$\frac{x+1}{(x-1)(x^2+1)^2} \equiv \frac{1}{2(x-1)} - \frac{x+1}{2\,(x^2+1)} - \frac{x}{(x^2+1)^2}$$

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Methods of converting improper fractions

When the degree of *p(x)* is equal to or greater than *q(x),* there will be no common factor. In this case, $\frac{p(x)}{q(x)}$ is considered improper fraction, for example $\frac{3x^2+1}{x^2-1}$ is an improper fraction. This is a polynomial expression. This kind of expression can be expressed as the sum of proper fractions, for example $\frac{3x^2+1}{x^2-1} = 3 + \frac{4}{x^2-1}$. By following the above process, we can express the proper fraction in partial fraction.

Example: Express $\frac{x^3+1}{x^2+1}$ into partial fractions

Solution: Tell what will be quotient and remainder if you divide the above fraction? Match your answer with quotient $= x$ and remainder $= -x+1$.

So, $\frac{x^3+1}{x^2+1} = x + \frac{-x+1}{x^2+1} = x - \frac{x-1}{x^2+1}$

Here, $\frac{x-1}{x^2+1}$ is a partial fraction

Team project

According to the instruction of the teacher, the students will divide themselves into several groups and accomplish the following tasks.

Instruction:

1. The teacher will write 5 polynomial expressions (2 linear polynomials, 2 quadratic polynomials, 1 cubic polynomial) for every group.
2. Every group will show the geometric form of the expression on graph papers.
3. Five natural objects similar to the geometric forms have to be attached with the graph papers.
4. Factor the quadratic and cubic polynomials given by the teacher and present them on poster paper.
5. According to the instructions of the teacher, present everything on poster paper in a specific date.

Exercise

1. Form polynomial expressions from three real life examples.
2. Give examples of polynomial expressions according to the following instructions.
   i) one variable, three dimensional, binomial
   ii) one variable, three dimensional, quadrimonial
   iii) two variables, three dimensional, binomial
   iv) two variables, three dimensional, trinomial
   v) four variables, cyclic, four dimensional
3. Give example:
   i) homogeneous, symmetric, cyclic polynomial
   ii) homogeneous, symmetric, polynomial; not cyclic
   iii) homogeneous, cyclic polynomial; not symmetric
   iv) symmetric, cyclic polynomial; not homogeneous
4. i) Divide $x^4 - 3x^2 + 1$ by $2x^2 - 3$
   ii) Divide $5x^3 - 3x - 2$ by $3x - 2$ and find out the remainder. Prove the appropriateness of the remainder through the remainder theorem.
5. Which of the following expressions are real prime expressions? Factor the expressions that are not real prime expressions.
   i) $x^2 - 5x - 14$ ii) $x^2 - 5x + 2$ iii) $2x^2 + 3x + 1$ iv) $3x^2 + 4x - 1$
6. Factor the following expressions.
   i) $x^3 - 5x + 4$ ii) $x^3 - 3x^2 + 3x - 2$ iii) $x5 - 16xy^4$
7. The length of a cubic reservoir is the inverse multiple of the length of another cubic reservoir. The sum of the length of the two reservoirs is 3 feet. What will be the sum of their volume?
8. Express the following fractions in partial fractions.
   i) $\dfrac{x+1}{(x-1)^2(x^2+1)^2}$, ii) $\dfrac{x^3+1}{x^2+1}$

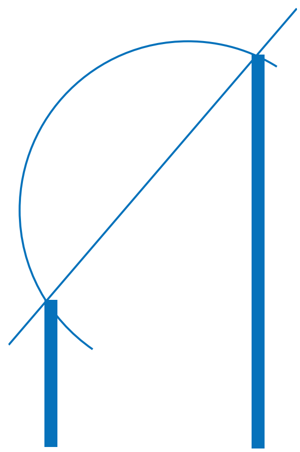
Instruction for the students

In order to understand the significance and usage of polynomial expressions, the students have to complete individual, pair and group tasks with great importance. The students have to create a problem and find out the solution on their own. This is how the students will have proper understanding

System of Equations in Real World Problems

You can learn from this experience-

- Linear equation in two variables
- Solving quadratic equations in one variable
- Solving quadratic equations with graphs
- Solving system of linear equation in two variables and quadratic equation in one variable



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System of Equations in Real World Problems

Equations are one of the most important topics in algebra for solving mathematical problems. You have already got the concept of linear equations in previous classes. You also know how to solve linear equations in one variable. Moreover, you can solve various practical problems that you have to deal with in your daily life by forming some linear equations, right? Do all the equations you have to form have one variable? Or in some cases there may be two or more variables. Let us try to understand the matter by solving some real problems.

Formulation and solving linear equations in two variables

Try to solve the problems in Table 5.1 below by formulating equations:

Table 5.1

Problem	Equation	Solution
1. How many different squares can be made with 20 matchsticks?		
2. The age ratio of Lily and her brother is 3:4; If the total age of the two is 21 years, how old is Lily?		
3. Setu buys two erasers and a pencil from the shop for Tk 20 The shopkeeper didn't tell her how much each of them costs. Can you tell how much the price of each thing can be?		

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Observation:

a) To solve problem (1) we formed equation with ☐ variables and found ☐ values of the variables, by which the equation is satisfied.

b) To solve problem (2) we formed equation with ☐ variables and found ☐ values of the variables, by which the equation is satisfied.

c) To solve problem (3) we formed equation with ☐ variables and found ☐ values of the variables, all of which satisfies the equation. In this case there was no definite solution for the variables.

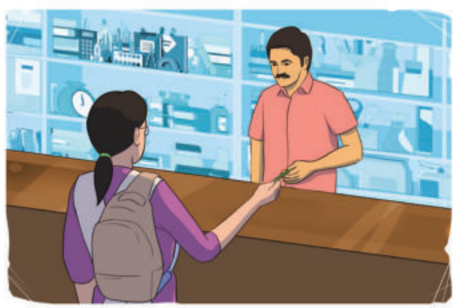
Solution of Setu's Problem

To solve problem number 3 in the above table, Setu is also trying hard on her own. Like you, she first assumes that an eraser costs Tk x and a pencil costs Tk y. The problem was then analyzed to obtain the following equation in two variables. $2x + y = 18$ ... ... (i)

Now make a table like Table 5.2 and verify the left and right sides of the obtained equation with different values of x and y

Table-5.2

Value of x	Value of y	Value of left side $(2x + y)$	Right side
1	16	$(2 \times 1) + 16 = 18$	18
2	14	$(2 \times 2) + 14 = 18$	18
3	12	$(2 \times 3) + 12 = 18$	18
4	10	$(2 \times 4) + 10 = 18$	18
…	…	…	18

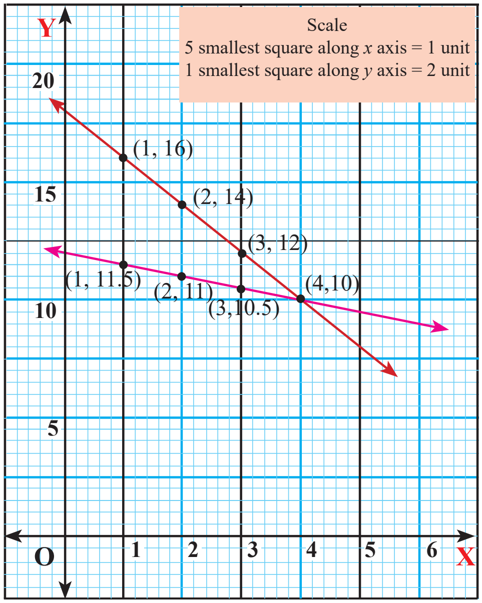


We can see that, the left and right sides of the equation match for many values of x and y. That is, the equation has numerous solutions. The solutions are: (1, 16), (2, 14), (3, 12), (4, 10),……. This means that the purchase price of an eraser and a pencil can vary. But how is that possible? She again bought an eraser and two pencils from the store to analyze the problem more deeply. This time the shopkeeper took a total of Tk

24 from Setu. As before, taking the price of an eraser as Tk x and the price of a pencil as Tk y, Setu analyzed the problem and obtained the equation in two variables $x + 2y = 24$......(ii). Now make the following table (Table-5.3) and verify the validity of the left and right sides of the obtained equation with differ values of x and y

Table-5.3

Value of x	Value of y	Value of left side $(2x + y)$	Right side
1	11.5	$1 + 2(11.5)$	24
2	11	$2 + 2(11)$	24
3	10.5	$3 + 2(10.5)$	24
4	10	$4+ 2(10)$	24
…	…	…	24



It also shows that the left and right sides of the equation are true for many values of x and y. That is, the equation has numerous solutions and the solutions are: (1, 11.5), (2, 11), (3, 10.5), (4, 10),…… This means, again, the purchase price of an eraser and a pencil can vary.

Setu consulted her teacher about this. Her teacher advised to solve the problem by considering the two equations together as a system. She said, you have known in previous class that such equations are equations of one straight line and such straight lines can be drawn by plotting points on graph paper. So, you can represent the two equations by two straight lines on a graph paper. Maybe, two straight lines can intersect at some point. If the straight lines intersect at a point, knowing the x and y values of that point will be your desired solution.

As suggested by her teacher, Setu first draws two straight lines on a graph paper. As per the drawing the straight lines intersect at the point (4, 10). At the point of intersection, $x = 4$ and $y = 10$. Setu understands that an eraser costs Tk 4 and a pencil costs Tk 10.

As you must have understood, some real-world phenomenon can thus be expressed by means of two variables and two linear equations. Such a system is called simultaneous linear equations in two variables and the point of intersection of the equations is called the solution of the simultaneous linear equations.

Pair Work:

a) Suppose you do not know the length and width of your rectangular classroom. The teacher said that the classroom's double of width is 10m longer than the length and in total 100m in perimeter. Your job is to form two equations with the length and width of the classroom as two variables and solve the two equations to find the area of the classroom. [You can solve by drawing straight lines on graph paper]

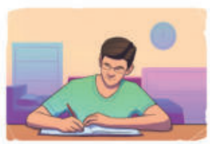
b) Now measure the length and width of the classroom by hand and find out the area of the floor. Then check the correctness of the area obtained from 'a'.

Rafi and Sonia's problem and solution

Problem: Setu's friend Rafi buys 2 packets of pin and 3 pens at Tk 28 and Sonia buys 4 packets of Pin and 6 pens from the same shop at Tk 56. In this case neither Rafi nor Sonia knows the price of one packet of pin and one pen. Can't we solve Rafi and Sonia's problem like we did for Setu? Let's try it:

Solution: Suppose, the price of 1 packet of pin is Tk x and the price of 1 pen is Tk y.

$\therefore$ the equations are,



$2x + 3y = 28$ ....(i)

and $4x + 6y = 56$ ....(ii)

What type of equation would you call the two equations (i) and (ii)? Write in the blank space below with the reason.

Now graph the two equations on graph paper and find their point of intersection.

First calculate some points to graph the lines for equations (i) and (ii) .

From equation (i),

x	$y = \dfrac{28 - 2x}{3}$	(x, y)
2		
5	6	(5, 6)
14		

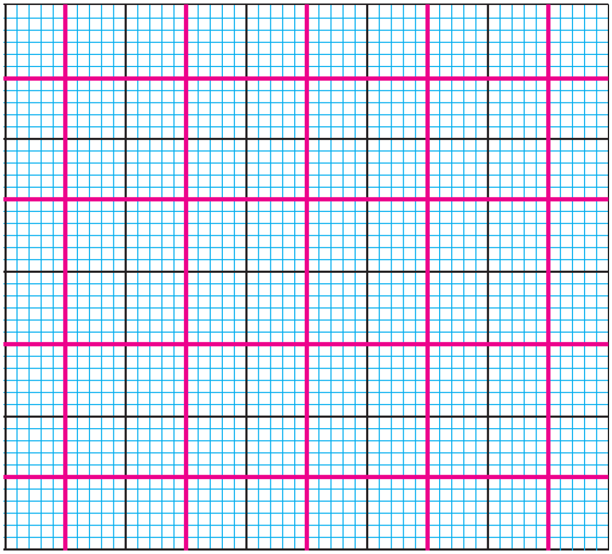
From equation (ii),

x	$y = \dfrac{56 - 4x}{6}$	(x, y)
2	8	(2, 8)
8		
11		

From equation (i) we obtain $(x, y) = (\quad , \quad), (5, 6), (\quad , \quad), (\quad , \quad)$

and from equation (ii) we obtain $(x, y) = (2, 8), (\quad , \quad), (\quad , \quad), (\quad , \quad)$

On the adjacent graph paper draw two straight lines (take the scale suitable to your convenience) by plotting the points obtained from equations (i) and (ii).



What do you see? Do two straight lines coincide? That is, every point on straight line (i) lies on straight line (ii).

So, every coordinate on the line (i) is also on the line (ii). If 1 packet of pin costs Tk, 1 pen will cost Tk. Again if 1 packet of pin costs Tk 2, 1 pen will cost Tk 8 and so on. In this case, how many solutions of equations (i) and (ii) have been found?

Individual task:

Khushi buys 2 poster papers and 3 sign pens for Tk 30. Dola buys 4 poster papers and 6 sign pens for Tk 50 of the same price from the same shop.

a) Formulate the equation and draw the graph.

b) Explain whether the solution of the two equations can be obtained from the diagram.

c) Express your opinion about the price of a poster paper and a sign pen.

Consistency of two simultaneous linear equations

Two simultaneous linear equations may have one solution, multiple solutions, or no solutions. So, it will be helpful if we can figure out in advance whether there is a solution or not. Let us try to find out the conditions about existence of solution through various

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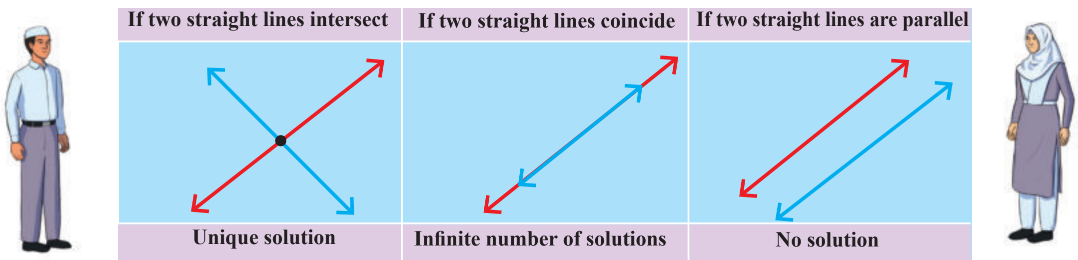
Geometric Observations

Solving simultaneous linear equations in two variables by graphing gives the following conditions.

a) When two straight lines intersect at a point, the equations have only one common solution.

b) When two straight lines coincide, there is only one straight line and the equations have infinitely many common solutions.

c) When two straight lines don't intersect and parallel to each other, the equations have no common solution.



Algebraic observations

In general, for the equations $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$, comparing the coefficients of x, y and the constant terms we can determine the solvability of them (Table- 5.4).

Table- 5.4

Simultaneous linear equations of two variables	Comparison of ratios	Location of lines on the graph	Consistent / Inconsistent	Algebraic decision
$a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Two intersecting straight lines	Consistent	Only one common solution
	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Two coinciding straight lines	Consistent	Infinitely many common solution
	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Two different straight lines which are parallel to each other	Incoensistent	No common solution

Pair Work:

For the given equations, write another linear equation of two variables which satisfies the given condition.

Given linear equation	Condition		
	Only one common solution	Infinitely many common solution	No common solution
$2x + 3y = 7$			
$y - 4x = 2$			
$-2x + 5y = 8$			
$3x - \frac{6}{5}y = 2$			

Brain storm

1. For what value of p there will be only one common solution to the equations $3x - 4y = 1$ and $9x + py = 2$?
2. For what value of r there will be no common solution to the equations $rx + 2y = 5$ and $(r + 1)x + 3y = 2$?
3. For what value of k there will be infinitely many common solutions to the equations $kx + 6y = k$ and $(k - 1)x + 4y = 5 - k$?
4. For what values of a and b there will be infinitely many common solutions to the equations $3x - (a + 1)y = 2b - 1$ and $5x + (1 - 2a)y = 3b$?

Methods of solving simultaneous linear equations in two variables

We can solve simultaneous linear equations in two variables mainly by geometric and algebraic methods. Let's know the methods.

Geometric Method	Algebraic Methods
Graphical Method	• Substitution Method
	• Elimination Method
	• Cross Multiplication Method

Solving by Graphical Method

By now you have experienced how to solve simultaneous linear equations in two variables by graphing them geometrically. As you have already seen, each linear equation has a straight line. Also, the coordinates of each point on the straight line satisfy the corresponding equation. Therefore, the coordinates of two or more points are required to specify the graph of a linear equation. Let us try to find the common solution of the following two linear equations with two variables by graphical method

Example: Solve the following two linear equations in two variables by graphical method.

$4x - y = 5$………(i)

$7x - 4y = 2$………(ii)

Instruction

Let us check the solvability: $\frac{4}{7} \neq \frac{1}{4}$

$\therefore$ The equation system is consistent and there is only one common solution.

Solution: To draw the graph of equations (i) and (ii) determine coordinates of three points for each equation.

From equation (i) we get, $y = 4x + 5$

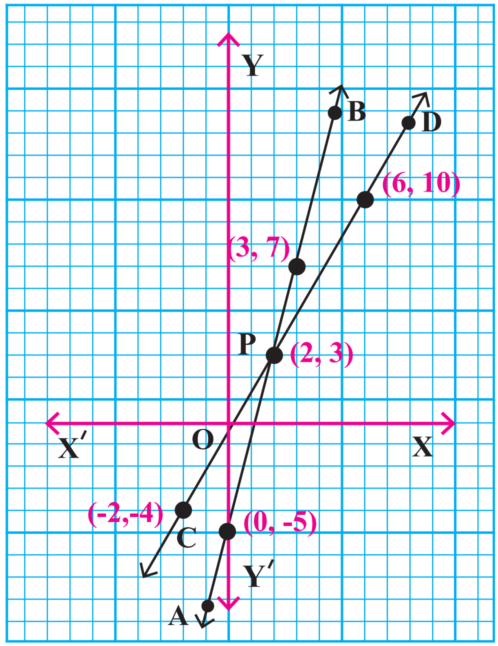
x	2		0
$y = 4x + 5$		7	-5

Again, from equation (ii) we obtain,

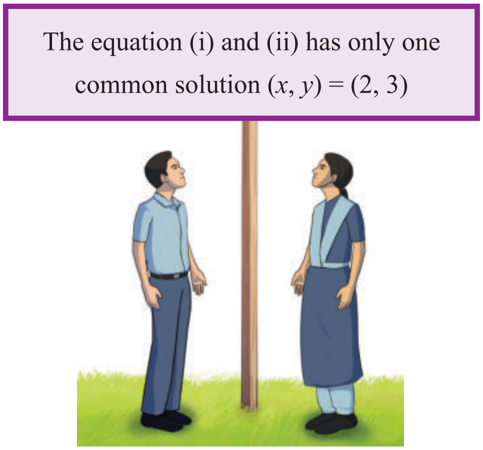
$y = \frac{7x - 2}{4}$

x		-2	6
$y = \frac{7x-2}{4}$	3		

Plot the points on a graph paper and draw the lines



From the graph we can see that, the equation (i) [Line AB] and the equation (ii) [Line CD] intersects at a common point. We calculate the coordinates of the point of intersection P to be (2, 3).



Individual task:

Sketch the graph and solve the following pairs of equations that are solvable. Write at least three solutions if they have infinite solutions:

i) $4x - 3y = 6$	ii) $4x + 3y = 20$	iii) $\frac{x}{5} + \frac{y}{4} = 23$	iv) $3x - \frac{2}{y} = 5$
$4y - 5x = -7$	$8x + 6y = 40$	$\frac{x}{4} + \frac{y}{5} = 22$	$x + \frac{4}{y} = 4$

Solving by Substitution Method:

In this method we can solve simultaneous lineat equations in two variables by following the steps below:

Step-1: Express the value of one of the two variables from any equation in terms of the other.

Step-2: In the other equation, substitute the value of the variable obtained in step-1 to make an equation of single variable and solve it.

Step-3: Substitute the already calculated value of the variable from step-2 and determine the value of the other variable.

Instruction

Let us check solvability: $\frac{1}{2} \neq -\frac{3}{1}$

$\therefore$ System is consistent and it has unique solution

Example: Let us try to find the common solution of the following two equations by solving the system of equations with two variables by substitution method.

$x + 3y = 16$.........(i) $2x - y = 4$.........(ii)

Step - 1	Step - 2	Step - 3	Solution
From equation (ii) we get, $2x - y = 4$ $\therefore\ y = 2x - 4$...(iii)	Substituting the value of y in equation (i) we obtain, $x + 3(2x - 4) = 16$ or, $x + 6x - 12 = 16$ or, $7x = 16 + 12$ $\therefore\ x = 4$	Putting $x = 4$ in equation (iii) we get, $y = 2.4 - 4$ or, $y = 8 - 4$ $\therefore\ y = 4$	$x = 4$ and $y = 4$ So, the solution is $(x, y) = (4, 4)$

Individual task:

Solve the pairs of equations which are solvable by substitution method and verify that the values of the variables found in the solution satisfy the equations.

i) $2x + 3y = 32$, $11y - 9x = 3$

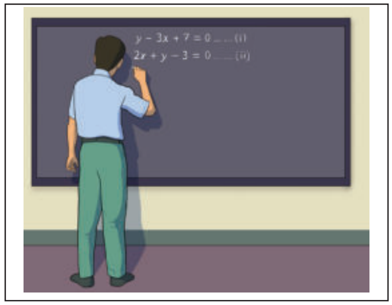
ii) $8x + 5y - 11 = 0$, $3x - 4y - 10 = 0$

iii) $\frac{2}{x} + \frac{5}{y} = 1$, $\frac{3}{x} + \frac{2}{y} = \frac{19}{20}$

iv) $x + y = p + q$, $px - qy = p^2 - q^2$

Example-2: Rafi wrote two equations $y - 3x + 7 = 0$ and $2x + y - 3 = 0$ on the board

Setu



Equations written by Rafi are: :

$y - 3x + 7 = 0$......(i)

$2x + y - 3 = 0$......(ii)

To solve, we follow the step

:

Instruction

Let us check the solvability: $-\frac{3}{2} \neq \frac{1}{1}$

$\therefore$The equation system is consistent and there is only one common

Step - 1	**Step - 2**	**Step - 3**	**Solution**
From (i) we obtain, $y - 3x + 7 = 0$ $\therefore y = 3x - 7$...(iii) Again, from (ii) we get, $2x + y - 3 = 0$ $\therefore y = -2x + 3$...(iv)	Putting the value of y from (iii) into the equation (iv),we get $3x - 7 = -2x + 3$ or, $3x + 2x = 3 + 7$ or, $5x = 10 \ \therefore x = 2$	Putting $x = 2$ in equation (iii), $y = 3.2 - 7$ or, $y = 6 - 7$ $\therefore y = -1$	x=2 and y=-1 So, required solution is, $(x, y) = (2, -1)$

Pair work:

Solve the following system of equations by method substitution:

i) $4x - 3y = 16$ $5y + 6x = 62$

ii) $2x + y - 8 = 0$ $3x - 2y - 5 = 0$

iii) $\frac{x}{3} + \frac{y}{4} = 1$ $2x + 4y = 11$

iv) $x + \frac{2}{y} = 7$ $2x - \frac{6}{y} = 9$

Solving by Elimination Method

By elimination we can solve simultaneous linear equations in two variables by following these steps:

Step-1: One equation or both equations must be multiplied by suitable numbers such that after multiplication the absolute values of the coefficients of any one of the variables in both equations are equal.

Step-2: Add or subtract the two equations as needed to eliminate the coefficient equalizing variable whose coefficients are made equal in step-1. Then solve the new equation after addition or subtraction to find the value of the existing variable.

Step-3: Substitute the obtained value of the variable in step-2 in either of the given equations to find the value of the other variable.

In the math class, the teacher said, let's play a fun game today. The game is to solve mathematical puzzles or problems created by one person. The condition is that the puzzle or problem must be such that the solution requires the formulation of simultaneous linear equations in two variables. Then solve the equations by eliminating one variable from the equations. After listening to the teacher, Setu asked Rafi to solve the following problem.

For a fraction, the numerator is x and the denominator is y. If we add 7 with the numerator the value of the fraction is 2. Again, if we subtract 2 from the denominator the value of the fraction is 1. What is the original value of the fraction?

Rafi first reads the problem carefully. Then he formulates the two equations as follows:

Step - 1	Step - 2	Step - 3	Solution
$\frac{x+7}{y} = 2$ or, $x + 7 = 2y$ $\therefore x - 2y = -7 \ldots$(i) and $\frac{x}{y-2} = 1$ or, $x = y - 2$ $\therefore x - y = -2 \ldots$(ii)	In equations (i) and (ii), the coefficients of x are equal and are of the same sign. So he subtracts equation (ii) from equation (i). That is, $x - 2y = -7$ $x - y = -2$ $- \quad + \quad +$ $-y = -5$ $\therefore y = 5$	Now in equation (ii), he puts $y = 5$ to get, $x - 5 = -2$ or, $x = -2 + 5$ $\therefore x = 3$	$x = 3$ and $y = 5$ So, required fraction $= \frac{3}{5}$

In this way, we can call the method to find out the solution by eliminating or removing any one variable of the simultaneous linear equations in two variables to be elimination method.

Individual task:

Solve the following systems of linear equations using elimination method:

i) $2x - 5y = 3$, $x + 3y = 1$

ii) $6x - y - 1 = 0$, $3x + 2y - 13 = 0$

iii) $\frac{x}{2} + \frac{x}{3} = 8$, $\frac{5x}{4} - 3y = -3$

iv) $ax + by = c$, $a^2x + b^2 y = c^2$

Solving by Cross Multiplication Method

Let's try to find out how this method determines the common solution of simultaneous linear equations in two variables.

First consider the following two equations:

$$a_1x + b_1y + c_1 = 0 \ldots\ldots\ldots \text{(i)}$$

$$a_2x + b_2y + c_2 = 0 \ldots\ldots\ldots \text{(ii)}$$

First, we want to determine the value of variable x from the equations (i) and (ii). Hence the variable y in equation (i) and (ii) must be eliminated. For this, multiply equation (i) by b_2 and equation (ii) by b_1:

$$a_1b_2x + b_1b_2y + b_2c_1 = 0 \ldots\ldots\ldots\ldots \text{(iii)}$$

$$a_2b_1x + b_1b_2y + b_1c_2 = 0 \ldots\ldots\ldots\ldots \text{(iv)}$$

Now, subtracting (iv) from (iii) we get,

$$(a_1b_2 - a_2b_1)\, x + b_2c_1 - b_1c_2 = 0$$

$$\text{or, } (a_1b_2 - a_2b_1)\, x = b_1c_2 - b_2c_1$$

$$\therefore \frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Again, we want to determine the value of variable y from the equations (i) and (ii). Hence the variable x in equation (i) and (ii) must be eliminated. For this, multiply equation (i) by a_2 and equation (ii) by a_1:

$$a_1a_2x + a_2b_1y + c_1a_2 = 0 \ldots\ldots\ldots \text{(vi)}$$

$$a_1a_2x + a_1b_2y + c_2a_1 = 0 \ldots\ldots\ldots \text{(vii)}$$

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Now, subtracting (vi) from (vii) we get,

$$(a_2b_1 - a_1b_2)y + c_1a_2 - c_2a_1 = 0$$

$$\text{or, } (a_2b_1 - a_1b_2)y = c_2a_1 - c_1a_2$$

$$\text{or, } -(a_1b_2 - a_2b_1)y = -(c_1a_2 - c_2a_1)$$

$$\therefore \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \ldots\ldots\ldots\ldots\ldots\ldots \text{(viii)}$$

Comparing equations (v) and (viii),

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}$$

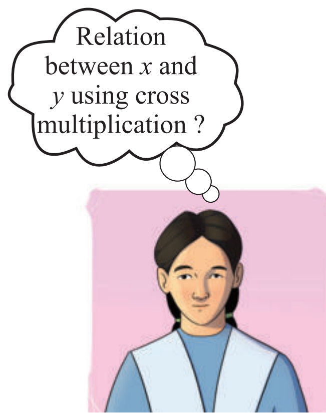
This method of finding values of x and y from this type of relation is called cross multiplication method.

From the above relation of x and y we can write,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \quad \therefore x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

and $\dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1} \quad \therefore y = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

Solution of the given system is: $(x, y) = \left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$



From the system of equations

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

We can directly write,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

We can use the following technique to obtain the solution of the above equations directly:

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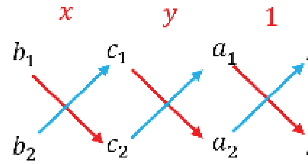
Equations	Relation between x and y	Technique to remember
$a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$	$\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c_2a_1}$ $= \dfrac{1}{a_1b_2 - a_2b_1}$	 Subtract the product of blue arrow sign digits from the product of red arow sign digits.

Problem: Planting trees

Every year a tree fair is held in the open field in front of Setu's school. One day after school ended, Setu and his friend Rahim went to the fair. Setu bought 4 guava saplings and 5 lemon saplings from a stall for Tk 310 to plant them in the empty space around their house and Rahim bought 3 guava saplings and 2 lemon saplings at the same rate and paid the salesperson a total of Tk 180. What are the prices of a guava tree sapling and a lemon tree sapling?

Solution: Let, the price of 1 guava sapling be Tk x and the price of 1 lemon sapling be Tk y . At first, we formulate the system of equations following the conditions:

$$4x + 5y = 310 \ldots\ldots\ldots \text{(i)}$$

$$3x + 2y = 180 \ldots\ldots\ldots \text{(ii)}$$

Instruction

Let us check the solvability: : $\frac{4}{3} \neq \frac{5}{2}$

$\therefore$ The equation system is consistent and there is only one common solution.

We want to find the common solution the system of equations by the method of cross multiplication. Therefore, we can write equations (i) and (ii) as follows:

$$4x + 5y - 310 = 0$$

$$3x + 2y - 180 = 0$$

Comparing the above equations with the following equations we obtain,

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$a_1 = 4$, $b_1 = 5$, $c_1 = -310$, $a_2 = 3$, $b_2 = 2$, $c_2 = -180$

Therefore, solving by cross multiplication method we get,

or, $\dfrac{x}{5 \times (-180) - 2 \times (-310)} = \dfrac{y}{(-310) \times 3 - (-180) \times 4} = \dfrac{1}{4 \times 2 - 3 \times 5}$

or, $\dfrac{x}{-900 + 620} = \dfrac{y}{-930 + 720} = \dfrac{1}{8 - 15}$

or, $\dfrac{x}{-280} = \dfrac{y}{-210} = \dfrac{1}{-7}$

or, $\dfrac{x}{40} = \dfrac{y}{30} = \dfrac{1}{1}$ [multiplying by -7]

or, $\dfrac{x}{40} = \dfrac{1}{1}$ again, $\dfrac{y}{30} = \dfrac{1}{1}$

$\therefore x = 40$ $\qquad \therefore y = 30$

$\therefore$ Price of a guava sapling is Tk 40 and price of a lemon sapling is Tk 30.

Individual task:

a) Solve the following system of equations by method of cross multiplication:

i) $5x - 2y = 32$
$4x - y = 28$

ii) $7x - 3y - 31 = 0$
$9x - 5y - 41 = 0$

iii) $x + 5y = 36$
$\dfrac{x + y}{x - y} = \dfrac{5}{3}$

b) Setu's reading room is of rectangular shape. If the floor length of the room is increased by 2 m and the width by 3 m, the area increases by 75 sq m. But decreasing the length by 2 meters and increasing the width by 3 meters increases the area by 15 square meters. Find the area of the floor of the reading room of Setu by forming the equations and solving by cross multiplication method.

Team project

The teacher will divide all the students in the class into 3 groups. Now she will take 3 papers and write 2 simple equations on each paper. The she will write the same system of equations on each paper. Now she will fold the paper into 3 and write substitution method in one, subtraction method in one and multiplication method in one and give one paper to each group through lottery. Each team will complete the following tasks

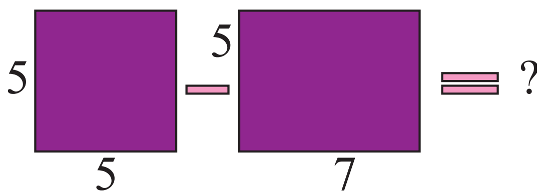
1. Check the consistency of the systems of equations.
2. Solve the system of equations by the method written on paper.
3. Check the correctness of the solution by drawing two straight lines on graph paper and finding the point of intersection.
4. All the activities will be presented on a poster paper and will be demonstrated as per the instructions of the teacher.

A linear equation in two variables and a quadratic equation in one variable

You are already familiar with linear equations in two variables. For example, $2x - 3y = 6$, this equation has two variables x and y each having the power one. Hence it is a linear equation in two variables. Now we will discuss quadratic equations in one variable.

Quadratic equations in one variable

Let's start with a fun quiz. Calculate and write the result of subtracting the area of the rectangle from the area of the square below



Surely you can write,

$5^2 - 7 \times 5 = -10$

Note that the length of a side of a square and a side of a rectangle are equal to each other, which is 5.

If we did not know the length of the sides of these two geometric shapes, could we have subtracted the areas so easily?

In that case, we would depend on variables. Let's assume that the length of the equal side in both areas is x.

Then the above equation becomes,

$$x^2 - 7x = -10$$

Can you tell what kind of equation it is? Only one variable x is used in the equation. So, it is a single variable equation in terms of variables. Again, the highest power of the variable x in the equation is 2. Hence, it is a quadratic equation if we consider the power of the variable. Thus, combining the two properties together, it is said to be a quadratic equation in one variable.

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Solution method of quadratic equation in one variable

To solve a quadratic equation, all terms of the equation should be brought to the left of the = sign and a 0 will come to the right, following the mathematical rules. On the left we get a quadratic polynomial of one variable. This quadratic polynomial must be factorized (methods of factoring are discussed in the chapter on polynomials). Then the value of the variable must be determined by taking the value of each factor as 0.

Solution by middle-term expansion

For the quadratic polynomial of one variable,

$$ax^2 + bx + c$$

middle-term expansion means that, expressing b in terms of two numbers d and e in such a way that, we get $d + e = b$ and $de = ac$.

Example: Solve the equation $x^2 - 7x = -10$.

Solution: $x^2 - 7x = -10$ can be written as,

$$x^2 - 7x + 10 = 0$$

Now expanding the middle-term on left we get,

$x^2 - 5x - 2x + 10 = 0$

or, $x(x - 5) - 2(x - 5) = 0$

or, $(x - 5)(x - 2) = 0$

Considering the value of each factor to be 0,

$(x - 5) = 0$ or, $(x - 2) = 0$

Therefore $x = 5$ or, $x = 2$.

Note here that the square has length 5 in the real problem from which the equation was derived.

Brain storm

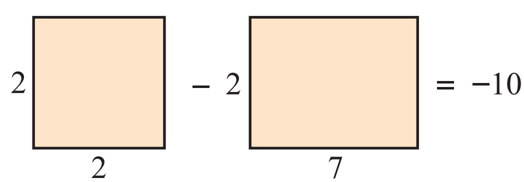
Express -7 in terms of two numbers in such a way that the sum is -7 and the product is 10.

Property of real numbers

$a.b = 0$ if and only if $a = 0$ or $b = 0$ is used here.

So where does $x = 2$ come from!

Interestingly, even if the length of the square is 2, the correct answer is obtained.



Note that by expanding the middle term we have easily factorized the left side of the equation $x^2 - 7x + 10 = 0$. But if the equation had $x^2 - 7x - 10 = 0$ would it be so easily solved by expanding the middle term? No, it couldn't be solved that easily (try it). There is a special way to solve all quadratic equations. Let us learn that method of solving quadratic equations.

General method of solving quadratic equations in one variable

he standard form of quadratic equations is:

$ax^2 + bx + c = 0$

where a, b, c are real numbers and $a \neq 0$

multiplying on both sides by $4a$ we get,

$4a^2x^2 + 4abx + 4ac = 0$ or,

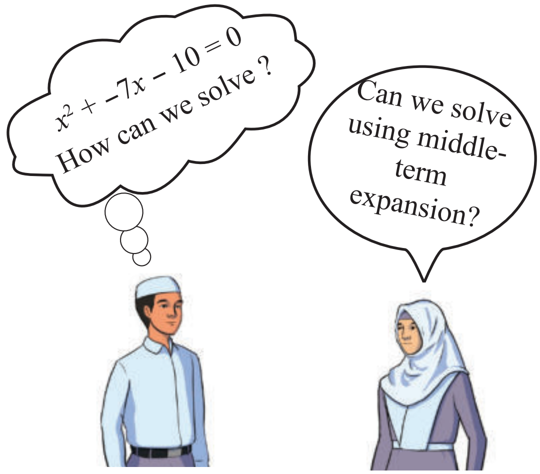
or, $(2ax)^2 + 2.2ax.b + b^2 - b^2 + 4ac = 0$

or, $(2ax + b)^2 = b^2 - 4ac$

or, $2ax + b = \pm\sqrt{b^2 - 4ac}$

or, $2ax = -b \pm\sqrt{b^2 - 4ac}$

$\therefore x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



Hence, the equation has two solutions or two values of x which are:

$x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$b^2 - 4ac$ is called the **discriminant** of the equation $ax^2 + bx + c = 0$. The value of the discriminant determines the nature of the roots.

- If $b^2 - 4ac = 0$ the roots are real and equal, and both are $x = -\dfrac{b}{2a}$
- If $b^2 - 4ac > 0$ and a perfect square then the roots are real, unequal and rational.
- If $b^2 - 4ac > 0$ and not a perfect square then the roots are real, unequal and irrational.
- If $b^2 - 4ac < 0$ there is no real root of the equation.

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Pair Work

Some equations are given below. Complete the following list by determining the nature of the roots of the equations.

Sl.	Equation	Discriminant $b^2 - 4ac$	Nature of the discriminant	Nature of roots
1	$2x^2 - 10x + 9 = 0$	$= (-10)^2 - 4.2.9$ $= 100 - 72$ $= 28$	$b^2 - 4ac > 0$ and not a perfect square.	Real, unequal and irrational.
2	$7x^2 - x + 2 = 0$			
3	$-5 + 7x + 6x^2 = 0$			
4	$-2x + 5 - 3x^2 = 0$			
5	$-14x + x^2 + 49 = 0$			
6		$= (-5)^2 - 4.3.4$ $=$		

As we mentioned earlier, solving the equation $x^2 - 7x - 10 = 0$ by expanding the middle-term is not easy. Let us now solve this equation using the general method.

Problem: Solve the equation $x^2 - 7x - 10 = 0$.

Solution: Comparing the equation $x^2 - 7x - 10 = 0$ with $ax^2 + bx + c = 0$ we get,

$a = 1$, $b = -7$, $c = -10$.

then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4.1(-10)}}{2.1} = \frac{7 \pm \sqrt{49 + 40}}{2}$$

$$\therefore x = \frac{7 \pm \sqrt{89}}{2}$$

Therefore, the roots are: $x_1 = \dfrac{7 + \sqrt{89}}{2}$ and $x_2 = \dfrac{7 - \sqrt{89}}{2}$

Individual task

Solve the following equations using the method you have learned. Write the roots in the blank spaces.

Sl.	Equation	Roots of the equation
1	$3x^2 - 5x + 1 = 0$	
2	$12x^2 - 11x + 2 = 0$	
3	$5x^2 - 8x + 4 = 0$	

Solving quadratic equations with graphs

Graphing the quadratic equation $ax^2 + bx + c = 0$ requires the value of x along with the value of y. Let $y = ax^2 + bx + c$. Then for values of x such that $y = 0$ i.e., the points at which the graph intersects the x-axis, all those values of x are solutions of the equation $ax^2 + bx + c = 0$.

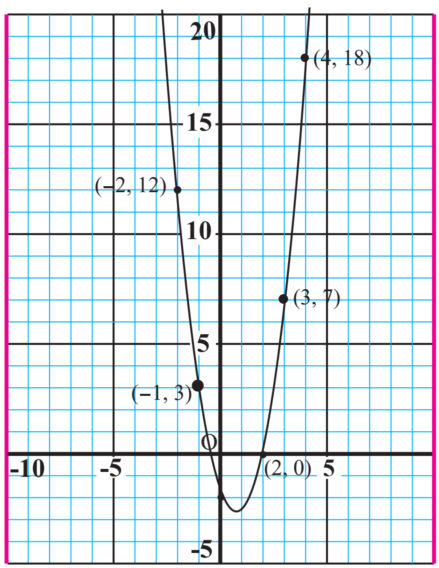
Example: Solve the equation $2x^2 - 3x - 2 = 0$ using graphs.

Solution: Suppose, $y = 2x^2 - 3x - 2$

We calculate some values of y for some values of x.

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y	12	7	3	0	–2	–3	–3	–2	0	3	7	12	18

On graph paper, take the length of the smallest square as one unit and plot the points to draw the graph.



Notice that the graph intersects the x-axis at the points $(-\frac{1}{2}, 0)$ and $(2, 0)$. The values of x at these two points are the solution of the given equation.

So, the required solution: $x_1 = -\frac{1}{2}$ and $x_2 = 2$

Individual task

Apply the formula to solve $2x^2 - 3x - 2 = 0$ and also solve it using graph. Check if both solutions are the same

A real-life problem and solution

Problem: Setu's uncle Mr. Hasan is a businessman. He bought a few packets of pens from a wholesale shop for Tk 50000. In another shop he bought pens of the same amount as he got the price Tk 2 less per packet of pens than the first shop and he got 25 more packets of pens. How many packets of pens did Mr. Hasan buy first and what was the price per packet of pens? What should be his selling price so that his total profit would be Tk 12000?

Solution: Can you represent the problem with equations? Let me help you. Suppose Mr. Hasan first bought x packets of pens. Now write the answers to the following questions in terms of x.

In first shop the price of each packet of pens = Tk $\boxed{}$

In second shop the price of each packet of pens = Tk $\boxed{}$

In second shop he bought = $\boxed{}$ packets

Price of pen bought in second shop = Tk $\boxed{}$

Total price of pens = Tk $\boxed{}$

According to the conditions,

$$\left(\frac{50000}{x} - 2\right)(x + 25) = 50000$$

$$\text{or, } (50000 - 2x)(x + 25) = 50000x$$

$$\text{or, } 50000x - 2x^2 + 50000 \times 25 - 50x = 50000x$$

$$\text{or, } -2x^2 + 50000 \times 25 - 50x = 0$$

$$\text{or, } 2x^2 + 50x - 50000 \times 25 = 0$$

$$\text{or, } x^2 + 25x - 25000 \times 25 = 0$$

This is a quadratic equation of one variable. Solving it using the general formula we get,

$$x = \frac{-25 \pm \sqrt{(25)^2 - 4 \times (-25000) \times 25}}{2} = \frac{-25 \pm \sqrt{(25)^2 + 4 \times 25000 \times 25}}{2}$$

$$= \frac{-25 \pm 25\sqrt{1 + 4000}}{2}$$

$$= \frac{-25 + 25\sqrt{1 + 4000}}{2}$$, [taking positive value as the number of packets can't be negative]

$$= \frac{25 \times (\sqrt{4001} - 1)}{2} \approx 778$$

Solving a system of simultaneous linear equation of two variables and quadratic equation of one variable

In practice, there are many problems that are easier to solve by converting them to linear equations in two variables and quadratic equations in one variable. An example of how to solve a mathematical problem is given first.

Example: Solve the following system of linear equation of two variables and quadratic equation of one variable.

$$y = 2x^2 - x - 3$$

$$x - 5y + 13 = 0$$

Solution: Suppose, $y = 2x^2 - x - 3$ …… (1)

Again given that, $x - 5y + 13 = 0$

or, $x + 13 = 5y$

or, $5y = x + 13$ or, $y = \frac{x + 13}{5}$ ……….(2)

We can write from (1) and (2),

$2x^2 - x - 3 = \frac{x + 13}{5}$

or, $10x^2 - 5x - 15 = x + 13$

or, $10x^2 - 5x - 15 - x - 13 = 0$

or, $10x^2 - 6x - 28 = 0$

or, $2(5x^2 - 3x - 14) = 0$

or, $5x^2 - 3x - 14 = 0$

or, $5x^2 - 10x + 7x - 14 = 0$

or, $5x(x - 2) + 7(x - 2) = 0$

or, $(x - 2)(5x + 7) = 0$

Therefore, $x - 2 = 0$ or $5x + 7 = 0$ $\quad \therefore x = 2$ or, $x = -\frac{7}{5}$

If $x = 2$, from equation (2) we obtain, $y = \frac{2 + 13}{5} = \frac{15}{5} = 3$

Again, if $x = -\frac{7}{5}$, from equation (2) we obtain,

$$y = \frac{-\frac{7}{5} + 13}{5} = \frac{\frac{-7 + 65}{5}}{5} = \frac{\frac{58}{5}}{5} = \frac{58}{25}$$

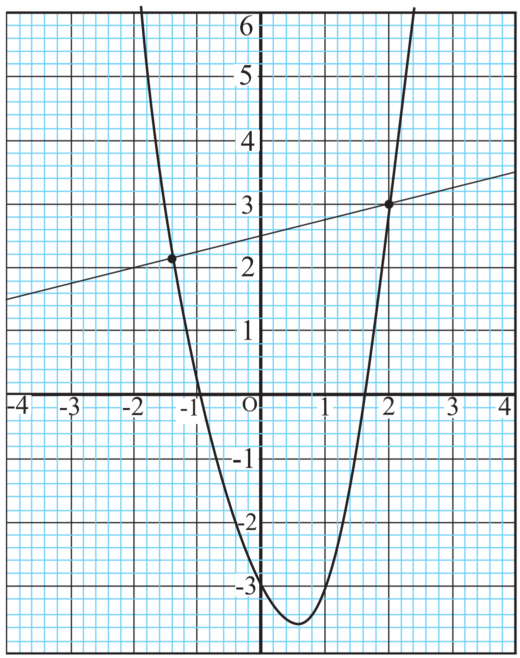
Required solution: $(x, y) = (2, 3), \left(-\frac{7}{5}, \frac{58}{25}\right)$

Solving using graphs

Given equations are

$$y = 2x^2 - x - 3$$

$$x - 5y + 13 = 0$$



Here, $x - 5y + 13 = 0$ is a linear equation and $y = 2x^2 - x - 3$ is a quadratic equation. You have learned to graph linear equations and quadratic equations. Using your experience, graph the two equations on the same plane. The graph is given in the figure on the side. Match it with your graph. It is observed from the figure that the two equations intersect at the points (2, 3) and $\left(-\frac{7}{5}, \frac{58}{25}\right)$. Both methods got us the same solution. Thus, the correctness of the solution is verified.

Team Project: Determinig supply according to demand

To make a factory profitable, it must produce goods equal to the demand of the consumer. This condition is called market equilibrium. Given below is the equation of supply in relation to the demand for the product produced in a factory.

$$q = p^2 - 2p + 44 \quad \dots\dots\dots \text{(i)}$$

$$p - q + 2 = 0 \quad \dots\dots\dots \text{(ii)}$$

Here, p is the price of the product and q is the quantity. Find the values of p and q for market equilibrium.

Work instruction:

1. Collect a poster paper, a graph paper and other necessary materials.
2. Solve algebraically. Describe the steps in the solution.
3. Plot the graphs of both equations (i) and (ii) on the same two dimensional coordinate axes on graph paper. Determine the point of intersection of the two graphs obtained.

4. Present your group's work methods and results on a poster paper or on the back of an old calendar. Consult with teacher if necessary.

   Writes arguments in favour of your group findings on poster paper.

   Match the solution [$p = 1, q = 3$ or, $p = 2, q = 4$]

Pair work

Following the instructions of class teacher, divide into several groups and solve the following system of equations algebraically. Then solve the equations graphically to prove that the solutions obtained both methods are the same. Write your group activities on posters and present them to the class.

$$y = x^2 - x - 2$$

$$x - 2y + 5 = 0$$

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Exercise

1. Compare the given systems of equations with $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$ and fill in the blanks.

Sl.	System of equations	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Comparison of ratios	Position on graph	Consistent/ Inconsistent	Algebraic decision
(i)	$x + 3y = 1$ $2x + 6y = 2$							
(ii)	$2x - 5y = 3$ $x + 3y = 1$							
(iii)	$2x - 4y = 7$ $x - 3y = -2$							
(iv)	$-\frac{1}{2}x - y = 0$ $x - 2y = 1$							

2. Sketch the graph and solve the following pairs of equations that are solvable. Write at least three solutions if they have infinite solutions:

i) $2x + y = 8$; $2x - 2y = 5$

ii) $2x + 5y = -14$; $4x - 5y = 17$

iii) $\frac{x}{2} + \frac{y}{3} = 8$; $\frac{5x}{4} - 3y = -3$

iv) $-7x + 8y = 9$; $5x - 4y = -3$

3. Solve using substitution method:

i) $7x - 3y = 31$; $9x - 5y = 41$

ii) $(x + 2)(y - 3) = y(x - 1)$; $5x - 11y - 8 = 0$

iii) $\frac{x}{a} + \frac{y}{b} = 2$; $ax + by = a^2 + b^2$

iv) $\frac{x}{14} + \frac{y}{18} = 1$; $\frac{x + y}{2} + \frac{3x + 5y}{2} = 2$

v) $p(x + y) = q(x - y) = 2pq$

4. Solve using elimination method:

i) $3x - 5y = -9$, $5x - 3y = 1$ ii) $\frac{x+1}{y+1} = \frac{4}{5}$, $\frac{x-5}{y-5} = \frac{1}{2}$ iii) $2x + \frac{3}{y} = 5$, $5x - \frac{2}{y} = 3$ iv) $ax + by = 1$, $bx + ay = \frac{2ab}{a^2 + b^2}$

5. Solve using cross multiplication method:

i) $3x - 2y = 2$, $7x + 3y = 43$ ii) $\frac{x}{2} + \frac{y}{3} = 8$, $\frac{5x}{4} - 3y = -3$ iii) $px + qy = p^2 + q^2$, $2qx - py = pq$ iv) $ax - by = ab$, $bx - ay = ab$

6. Apu has a rectangular vegetable garden. The perimeter of the garden is 120 meters. Doubling the width and subtracting 3 meters from the length the perimeter will 150 meters.
   a) The garden is enclosed on 3 sides and is open on one side along its length. How much money will it cost to enclose the empty side with a fence of tk. 10 Per meter?
   b) If organic fertilizer costs tk.7 per square meter, how much will Apu have to spend on fertilizer in total?

7. Find the nature of the roots of $x^2 - 3=0$ and solve it.

8. Solve $3x^2 - 2x - 1 = 0$ using formula. Again solve it with the help of graph and show that there is the same solution in both the method

9. Setu's mother keeps ducks and chickens at home.She bought 25 ducklings and 30 chickens for tk. 5000. If she had bought 20 ducklings and 40 chickens at the same rate, he would have spent tk. 500 less.

   a) What is the price of a duckling and a chicken?
   b) How much total profit will she make if she sells each duck at tk. 250 and each chicken at tk.1 60 after rearing for a few days?

10. Solve the following system of equations:

$y = x^2 - 2x - 3$

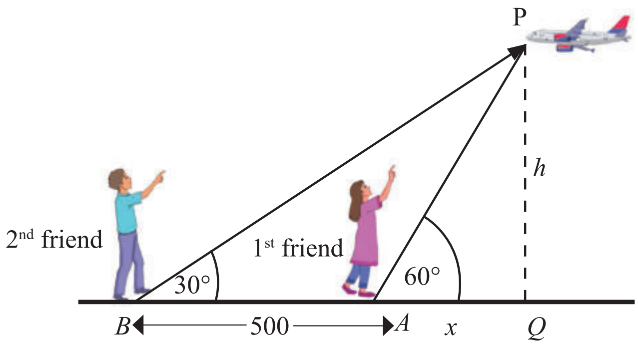
$x - 3y + 1 = 0$

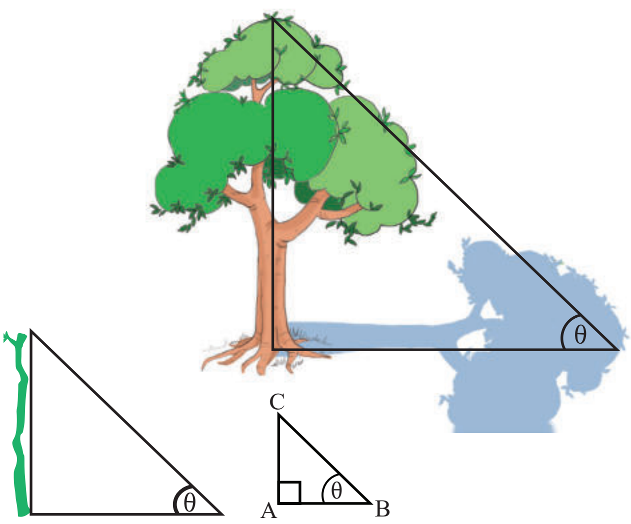
11. Construct 3 sets system of equations of two variables(one is linear and another is quadratic) as you like and solve it.

Trigonometry in Measurement

You can learn from this experience-

- Concept of trigonometry
- Tigonometric ratios.
- Values of different trigonometric ratios.
- Elevation and Depression Angle.
- Solution of real life problem related to distance and height.





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Trigonometry in Measurement

Suppose, one day, Abhi, Mita and Rina were discussing their studies sitting under a tree. Mita asked Avi if he could tell her the height of the tree. Abhi said, “Yes. I am going to measure the height of the tree right now.” Rina gave a condition that he could not climb up the tree and she proposed to learn how to measure the height of the tree without climbing up.

All of a sudden, Mita took their attention. She showed the shadow of the tree and they decided to find out a way of measuring the height of the tree by measuring the shadow. Abhi said, “Actually, the shadow is at the right angle to the tree. If we imagine some lines from the end point of the shadow to the top point of the tree, we will get a right angled triangle. Can it be of any use?”

Rina said, “Yes. We can use the Pythagoras theorem.”

Mita said, “If we have the length of two sides of a right angled triangle, we can find out the length of the third side through Pythagoras theorem. Here, we can only measure the length of the shadow of the tree. That would be the length of the base. But without measuring the length of the hypotenuse, we cannot find out the height. Maybe, we need a formula for that. Tomorrow, we will discuss this matter with our mathematics teacher.”

The next day, Abhi asked their teacher how they could measure the height of a tree without climbing up. The teacher told them to attend some classes on trigonometry. He also assured them that they would be able to solve the problem on their own after attending the classes.

1. Concept of Trigonometry

In the case of measuring, the right angled triangle plays an important role. You are aware of an interrelationship among the three sides of a right angled triangle. You learnt it in your previous class. Square on the hypotenuse is equal to the sum of the area of the square drawn on the other two sides. This relationship has been made based on the three sides of the triangle. But a right angled triangle has three angles along with the three sides. We can find out different relationships considering the angles and sides of the triangle. In ancient times, people used to solve different problems through the use of the ratio between the angles and sides of a triangle, for example how to measure the height of a tree without climbing up; how to measure the width of a river standing on one side of the river etc. These mathematical strategies have created a special branch of mathematics called trigonometry. The word ‘trigonometry’ originated from the Greek words ‘tri’ (three), ‘gon’ (side) and ‘metron’ (measure). The Egyptians used trigonometry for land measurement and engineering. Trigonometry deals with the problems related to triangles.

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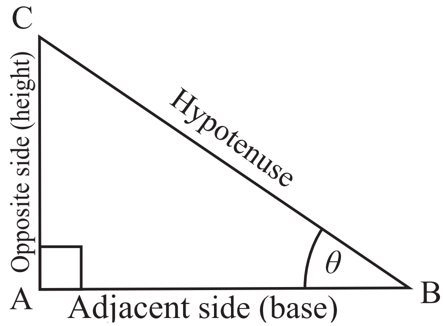
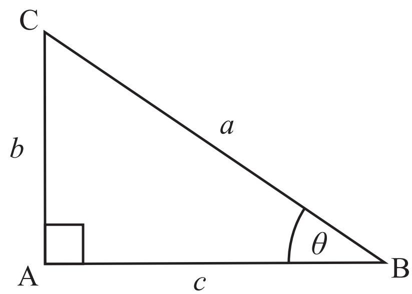
2. Introduction to Different Sides and Angles of Right Angled Triangle



In a right angled triangle, the opposite side of the right angle is called **hypotenuse**. There are two acute angles along with one right angle. The two acute angles are adjacent to the hypotenuse. One of the sides adjacent to the hypotenuse is called base and the other one is called height. The horizontal side parallel to the ground is the base and the vertical side located on the base is the height. If we turn the triangle and set the height parallel to the ground, the height will become the base and the base will become the height. So, the name of the sides depends on the position of the triangle. It is not a constant thing. Sometimes, it creates confusion. So, in order to avoid the confusion, we have to name the sides in reference to the angles. Suppose, we want to name the sides considering the angles adjacent to the base and hypotenuse. In this case, the base is regarded as the adjacent side and the height is regarded as the opposite side.



In geometric figures, for indicating the top points, capital letters (A, B, C etc.) are used and for indicating the sides, small letters (a, b, c etc.) are used. For indicating the angles, usually, Greek alphabets are used. Mathematicians from ancient Greece have been using these alphabets in geometry and trigonometry. Some of the alphabets are as follows:

Angle	α	β	γ	θ	δ
Name	alpha	beta	gamma	theta	delta

In the above figure of right angled triangle, the angle ∠ABC has been referred as θ.

Individual task:

Identify the hypotenuse, adjacent side and opposite side of the following triangles considering the angles θ and α.

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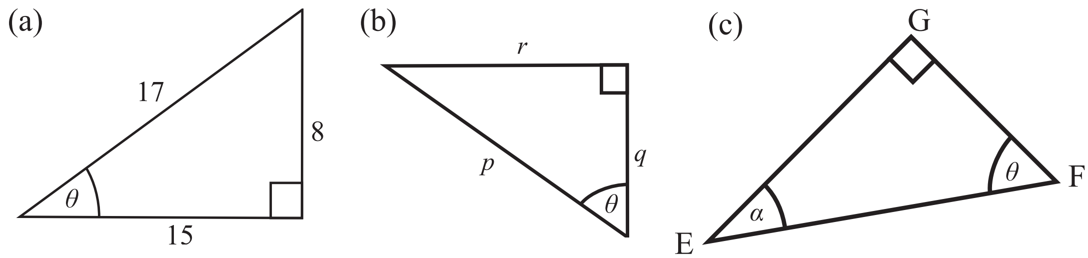

Name of right angled triangle	Acute angle	Hypotenuse	Opposite side	Adjacent side
a	θ	17		
b	θ			
c	θ			
d	α	EF		

3. The Sides of Different Sides of a Right Angled Triangle Considering the Interim Angle between the Hypotenuse and the adjacent side

Pair work:

Draw a right angled triangle in your notebook. You can draw the length of the sides according to your wish but the acute angle adjacent to the base has to be 30°. After drawing the triangle, measure the length of every side by using a scale and complete the following table.

(1) Adjacent side	(2) Opposite side	(3) Hypote-nuse	(4) $\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	(5) $\frac{\text{Opposite side}}{\text{Hypotenuse}}$	(6) $\frac{\text{Adjacent side}}{\text{Opposite side}}$	(7) $\frac{\text{Opposite side}}{\text{Adjacent side}}$	(8) $\frac{\text{Hypotenuse}}{\text{Adjacent side}}$	(9) $\frac{\text{Hypotenuse}}{\text{Opposite side}}$

Match the six ratios from number 4 to number 9 with your classmates. What have you noticed from the task? You have found the ratio of the sides in reference to a 30° acute angle. Though the sides have different lengths, the ratios are the same. If you find out the ratio of the sides in reference to any acute angle of a right angled triangle, we will see that the ratios are the same though the length of the slides are different. From this experiment, we can say that-

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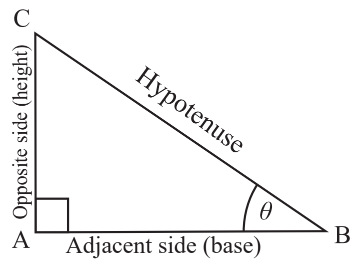
If the interim angle of base and hypotenuse of right angled triangles are equal, the ratio of the sides of those right angled triangles will be equal. If the measure of the of the interim angles are different, the ratio of their sides will be different too.

4. Naming Different Ratios in Reference to a Particular Angle

In a right angled triangle, the ratio of the sides in reference to a particular acute angle is equal. We have three sides: adjacent side, opposite side and hypotenuse. We can create ratios by using any two of the three sides. Do you have any idea about this? We can create six ratios in total. The ratios are as follows:

$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{\text{Hypotenuse}}{\text{Opposite side}}$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	$\frac{\text{Hypotenuse}}{\text{Adjacent side}}$	$\frac{\text{Opposite side}}{\text{Adjacent side}}$	$\frac{\text{Adjacent side}}{\text{Opposite side}}$

Mathematicians have set six different names for these six ratios. If the acute angle adjacent to hypotenuse and base is pointed by θ, the ratios will be: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$ and $\cot\theta$. The relationships these six ratios create with the sides are as follows:



$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{AC}{BC}$	$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{AB}{BC}$	$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{AC}{AB}$
$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite side}} = \frac{BC}{AC}$	$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent side}} = \frac{BC}{AB}$	$\cot\theta = \frac{\text{Adjacent side}}{\text{Opposite side}} = \frac{AB}{AC}$

These ratios are Trigonometric Ratios. Usually, the names of trigonometric names are written in short forms. Their full name are as follows:

Full name	**sine**	**cosine**	**tangent**	**cotangent**	**secant**	**cosecant**
Short form	sin	cos	tan	cot	sec	csc

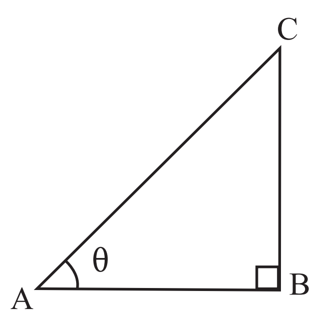
Pair work:

After analyzing the trigonometric ratio, try to find out whether all the other ratios can be expressed through $\sin\theta$ and $\cos\theta$. Two examples have been given below. Now, think and complete the table.

$\csc\theta, = \dfrac{\text{Hypotenuse}}{\text{Opposite side}} = \dfrac{1}{\dfrac{\text{Opposite side}}{\text{Hypotenuse}}} = \dfrac{1}{\sin\theta}$
$\tan\theta, = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{\dfrac{\text{Opposite side}}{\text{Hypotenuse}}}{\dfrac{\text{Adjacent side}}{\text{Hypotenuse}}} = \dfrac{\sin\theta}{\cos\theta}$
$\sec\theta = ?$
$\cot\theta = ?$

5. The Value of Trigonometric Ratio in Reference to Different Angles

5.1. For 45° angle



Suppose, $\triangle ABC$ is a right angled triangle. $\angle B = 1$ is a right angle and $A = 45°$.

So, $\angle C = 45°$ [$\because$ The sum of three angles of a triangle is equal to the sum of two right angle]

Then, $AB = BC$ [$\because$ The opposite sides of equal angles of a triangle are equal to each other]

Suppose, $AB = BC = a$.

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What we get by using the Pythagoras theorem is that

$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$

$\therefore$ AC = $\sqrt{2}$a

So, $\sin 45° = \sin A = \dfrac{\text{opposite side}}{\text{Hypotenuse}} = = \dfrac{BC}{AC} = \dfrac{a}{\sqrt{2}\,\text{a}} = \dfrac{1}{\sqrt{2}}$

In the same way, $\cos 45° = \cos A == \dfrac{\text{adjacent side}}{\text{Hypotenuse}} = \dfrac{BC}{AC} = \dfrac{a}{\sqrt{2}\,\text{a}} = \dfrac{1}{\sqrt{2}}$

Pair work:

Find out the value of the following trigonometric ratios. One sample has been provided.

$\csc 45° = \dfrac{1}{\sin 45°} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$
$\tan 45° = ?$
$\sec 45° = ?$
$\cot 45° = ?$

5.2. For 30° and 60° acute angles

In the adjacent figure, $\triangle ABC$ is an equilateral triangle. $\therefore A = B = C = 60°$ [Every angle of the equilateral triangle is 60°]

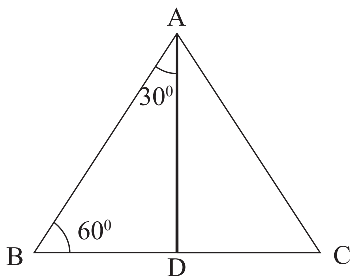
Draw a perpendicular AD on BC from the point A. The point D divides BC into two equal parts. So, BD = CD. Again the line AD divides the angle $\angle BAC$ into two equal angles.

So, $\angle BAD = \angle CAD = 30°$

Suppose, $AB = 2a$. So, $BD = \dfrac{1}{2}.BC = \dfrac{1}{2}.2a = a$ and

$AD = \sqrt{AB^2 - BD^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = \sqrt{3}a$

So, we can write:



$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2},\ \cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

Pair work

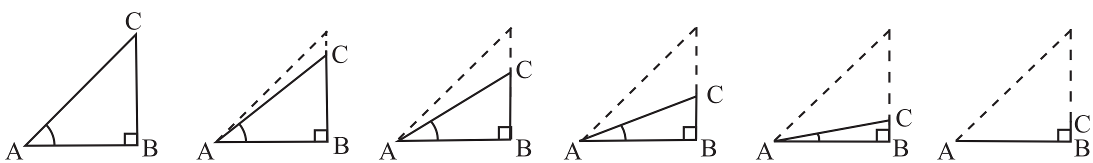
Find out the value of the following trigonometric ratios and show these to your teacher.

sin 30°, sin 60°, tan 30°, tan 60°, sec 30°, sec 60°, csc 30°, csc 60°, cot 30°, cot 60°

5.3. For 0° angle

We have learnt how to find out the trigonometric ratio of the angles 30°, 45° and 60°. Now, we will see what will be the shape of a triangle if its angle is 0° or 90°. We will also learn how to find out the value of the ratio of a triangle if its angle is 0° or 90°.

Suppose, △ABC is a right angled triangle. If the value of the angle ∠A of the triangle becomes smaller, the length of BC will get smaller too. The closer the value of ∠A gets to 0, the closer the length of BC gets to zero.



In the triangle △ABC, the value of $\sin A = \frac{\text{opposite side}}{\text{Hypotenuse}} = \frac{BC}{AC}$ will be close to 0. In this case, the length of AC will be nearly equal to AB.

Then, the value of $\cos A = \frac{\text{adjacent side}}{\text{Hypotenuse}} = \frac{AB}{AC}$ will be 1.

This idea helps us to define $\sin A$ and $\cos A$ when A = 0°. We can write:

$$\sin 0^\circ = 0 \text{ and } \cos 0^\circ = 1$$

Individual task:

Find out the value of tan 0°, cot 0°, sec 0° and csc 0° by using the value of sin 0° and cos 0° .

5.4. For 90° angle

When the angle ∠A in △ABC become larger, the length of AB gets smaller.

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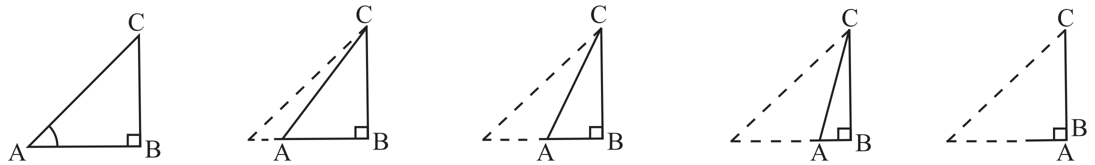


The closer the value of $\angle A$ gets to $90°$, the closer the length of AB gets to 0.

In the triangle $\triangle ABC$, the value of $\cos A = \frac{\text{adjacent side}}{\text{Hypotenuse}} = \frac{AB}{AC}$ will be close to 0.

Then, the value of $\sin A = \frac{\text{opposite side}}{\text{Hypotenuse}} = \frac{BC}{AC}$ will be close to 1.

This idea helps us to define cos A and sin A when $A = 90°$.

Then we can write:

$$\cos 90° = 0 \text{ and } \sin 90° = 1.$$

Individual task

Find out the value of tan 90°, cot 90°, sec 90° and csc 90° by using the value of sin 90° and cos 90° .

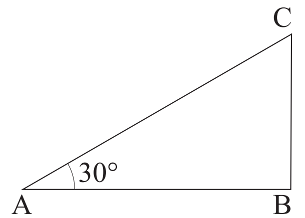
We can write the value of the trigonometric ratios in the following manner.

$\theta°$ / **ratios**	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined
cot	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
csc	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

We can solve many problems by using the above table.

Problem: In a right angled triangle $\triangle ABC$, $\angle A=30°$ and AB = 7cm. Find out the length of BC and AC.



Solution: From $\triangle ABC$, we get: $\tan A = \frac{BC}{AB}$.

$\therefore \tan 30° = \frac{BC}{7} \Longrightarrow \frac{1}{\sqrt{3}} = \frac{BC}{7} \Longrightarrow BC = \frac{7}{\sqrt{3}} = 4.04$ cm (Approx.)

Again, $\cos A = \frac{AB}{AC}$

$\therefore \cos 30° = \frac{7}{AC} \Longrightarrow \frac{\sqrt{3}}{2} = \frac{7}{AC} \Longrightarrow AC = \frac{14}{\sqrt{3}} = 8.08$ cm (Approx.)

Pair work

Make a problem like the above problem and give it to your pair to solve the problem. Check the solution by your teacher.

6. Finding out the Trigonometric Ratios in Reference to Different Angles through Calculator

You have found the trigonometric ratios in reference to a particular angle by using the rules of the right angled triangle. It is very difficult to find out the trigonometric ratio in reference to an angle. Luckily, we have scientific calculators or computers that can be used to find out the trigonometric ratios. For your practice purpose, use your calculator and find out the ratios given to you as pair work.

Pair work:

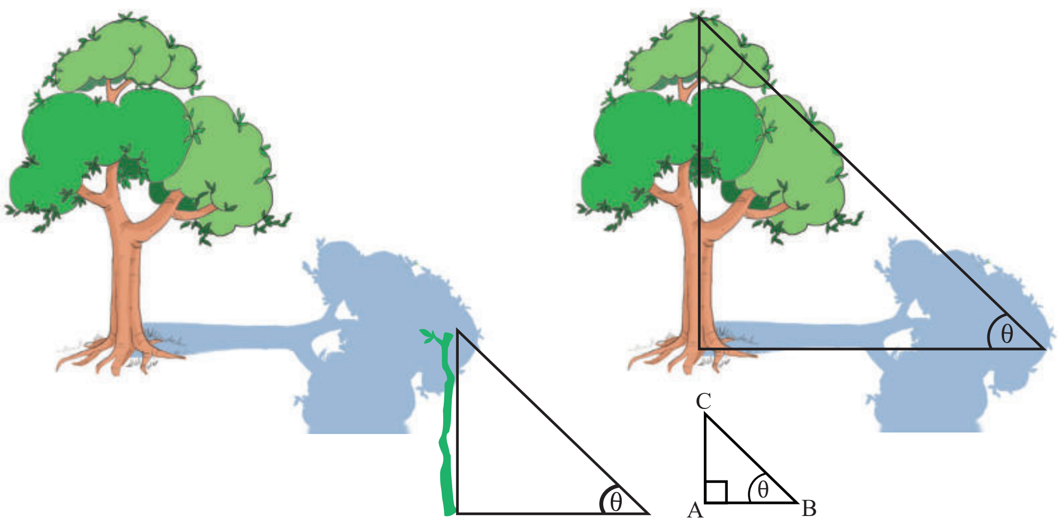
1) With the help of teacher, find out the ratio of 40°, 55°, 62°, 83° by using a scientific calculator or computer.

2) Find out the value of the following ratios. sin 32°,cos 36°,tan 52°,cot 61.5°,sec 72.6°,csc 15°

The mathematics teacher said to Rina, Abhi and Mita, “Now you have all the knowledge that you need to measure the height of a tree without climbing up. Let’s complete the following task with all the other students.”

Group project

All the students in the class will be divided into several groups. Each group will pick a stick or tree branch of different lengths according to their convenience. When the sun is in a reclining position, every group will go to a tree. After that they will set the stick or tree branch vertically on the ground and measure the length of the shadow. At the same time, they will measure the length of the shadow of the tree.



Now draw a triangle ΔABC like the above figure where the ratio of AC and AB is equal to the ratio of the length of stick and the length of the shadow of stick. It means:

$$\frac{AC}{AB} = \frac{\text{Length of the stick}}{\text{Length of the shadow of the stick}}$$

Suppose, $\angle ABC = \theta$. Then, $\tan\theta = \dfrac{AC}{AB} = \dfrac{\text{Length of the stick}}{\text{Length of the shadow of the stick}}$

Find out the value of $\tan\theta$ by using the measurement that you did earlier and write it down in your notebook.

Suppose, the height of the tree is h. The line from the top point to the ended point of the shadow of the tree makes an acute angle θ. So,

$$\tan\theta = \frac{h}{\text{Length of the shadow of the tree}}$$

That is,

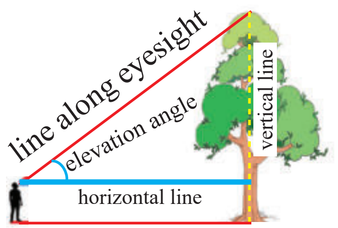
$$h = \tan\theta \times \text{Length of the shadow of the tree}$$

You have the value of tan θ and you know the length of the shadow of the tree. Now find out the value of h.

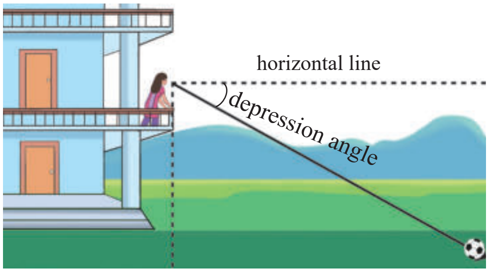
The groups you measured the shadow of the same tree will get the same or close value of h. If any group's value of h. doesn't match with others in case of the same shadow, there might be an error in the process. The group should try again.

7. Elevation and Depression Angle

Let's look into the adjacent figure. A person is looking at the top of the tree. If we imagine a line along with the person's eyesight, a geosynchronous line from the eye and a vertical line from the bottom to top point of the tree, we will get a right angled triangle. In this case, the angle situated in between the line along with the eyesight and the geosynchronous line from the eye is an elevation angle. If we know the measurement of this elevation angle and the length of one side of the imagined triangle, we can find out the lengths of other sides of the triangle by using trigonometric ratio.

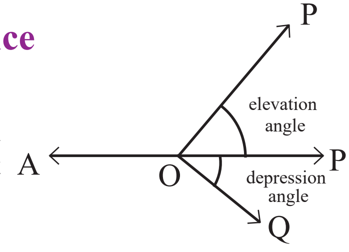


Now look into the adjacent figure. A child is looking down at an object from the balcony of the 1st floor. If we imagine a line along with the child's eye sight, a vertical line from the child's eye to the ground, a line from the object to the end point of the vertical line, we will get a right angled triangle. In this case, the angle situated in between the line along with the eyesight and the line imagined on the ground from the object to the end point of the vertical line is a depression angle. If we know the measurement of this elevation angle and the length of one side of the imagined triangle, we can find out the lengths of other sides of the triangle by using trigonometric ratio.



7.1. Elevation and Depression Angle in Reference to a Particular Point of a Particular Side

Suppose, AB is a line parallel to the ground. O is a point on AB. Two angles ∠POB and ∠BOQ has been drawn so that A, O, B, P and Q can be located on the same vertical plane.



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Here the point P is located above the line AB. So, the elevation angle P in reference to O is ∠POB.

Again, the point Q is located below the line AB. So, the depression angle of Q in reference to O is ∠QOB.

8.The Importance of Measuring Trigonometric Ratio

You have already understood the importance of trigonometric ratio. We use trigonometric ratios for different purposes in our daily life. We can measure the distance of an object by measuring the angle. We do not need to go near the object. This discovery was a revolution. So, we will concentrate more on acquiring the knowledge of trigonometry. We can solve many difficult problems with this knowledge.

9. Distance and Height: Real Life Problems and Their Solutions

Let's solve some real life problems by using the knowledge that we have gathered so far.

Problem 1: A ladder has been kept leaning against the edge of the roof of a house. The length of the ladder is 12 feet and the ladder has created a 45° angle with the land. What is the height of the roof from the ground?

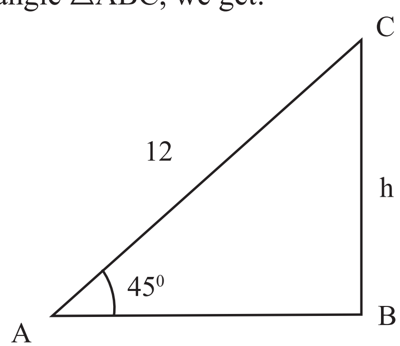
Solution: Suppose, the top point of AC is C and the point C is at the edge of the roof. So, the altitude drawn from the ground to the point C will be the height of the roof. According to the figure, BC = h (suppose), the height of the roof and the base AB have created a 45° angle. Therefore, ∠CAB = 45°. From the right angled triangle △ABC, we get:

$$\sin 45° = \frac{BC}{AC} = \frac{h}{12}$$

$$\Longrightarrow \frac{1}{\sqrt{2}} = \frac{h}{12}$$

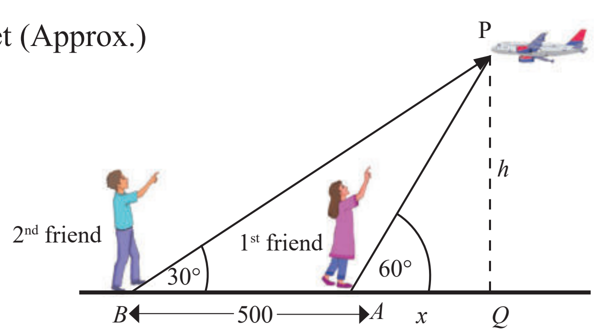
$$\Longrightarrow \sqrt{2}\,h = 12$$

$$\Longrightarrow h = \frac{12}{\sqrt{2}} = 8.49 \text{ feet (Approx.)}$$



So, the height of the wall is 8.49 feet (Approx.)

Problem 2: Two friends were standing 500 m away and they saw a plane flying over them. At a certain instant, the elevation angle of the plane from the 1st friend was 60° and the elevation angle of the plane from the 2nd friend was 30°.



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How high was the plane flying? If the plane passed over the 2nd friend 2 seconds later, what was the speed of the plane?

Solution: Suppose, the position of the 1st friend is A, the 2nd friend's position is B and the position of the plane is P. Suppose, the altitude imagined from P to ground is PQ = h and AQ = x.

From the right angled triangle ΔAPQ, we get: $\tan 60° = \frac{h}{x}$

$$\Longrightarrow \sqrt{3} = \frac{h}{x}$$

$$\Longrightarrow x = \frac{h}{\sqrt{3}} \text{ .... (1)}$$

From the right angled triangle ΔBPQ, we get: $\tan 30° = \frac{h}{x + 500}$

$$\Longrightarrow \frac{1}{\sqrt{3}} = \frac{h}{x + 500}$$

$$\Longrightarrow x + 500 = h\sqrt{3}$$

$$\Longrightarrow \frac{h}{\sqrt{3}} + 500 = h\sqrt{3} \quad \text{[Substituting the value of } x \text{ from the equation (1)]}$$

$$\Longrightarrow h + 500\sqrt{3} = 3h$$

$$\Longrightarrow 2h = 500\sqrt{3}$$

$$\Longrightarrow h = 250\sqrt{3}$$

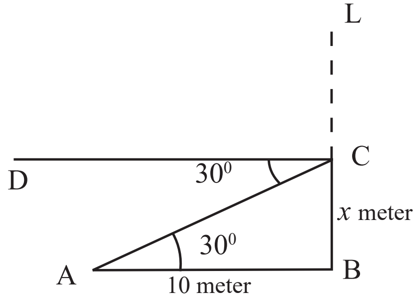
So, the plane is flying through an altitude of $250\sqrt{3}$ m.

We get from the equation (1): $x = \frac{h}{\sqrt{3}} = \frac{250\sqrt{3}}{\sqrt{3}} = 250$

The plane covers a distance of 500 + 250 = 750 m. in 2 seconds. So, the speed of the plane is 750 ÷ 2 = 375 m/s.

Problem 3: A pole broke in such a way that its broken part touched the ground 10 m from the base of the pole. The depression angle is 30°. So, what is the length of the pole?

Solution:



Let's assume, the length of the pole was BL= h m and the pole broke at the height of BC = x m. The broken part touched the ground AB = 10 m from the base of the pole. So, AC = CL.

Here the depression angle is $\angle ACD = 30°$. So, $\angle BAC = \angle ACD = 30°$ [Alternate angle]

According to the rule, AC = BL - BC = (h - x) m. So, we get from the triangle $\triangle ABC$,

$$\tan 30° = \frac{BC}{AB} = \frac{x}{10}. \quad \Longrightarrow x = 10 \tan 30° = 10 \times \frac{1}{\sqrt{3}} = \frac{10}{\sqrt{3}} \text{ m}$$

Aganin, $\cos 30° = \dfrac{AB}{AC} = \dfrac{10}{h - x}$

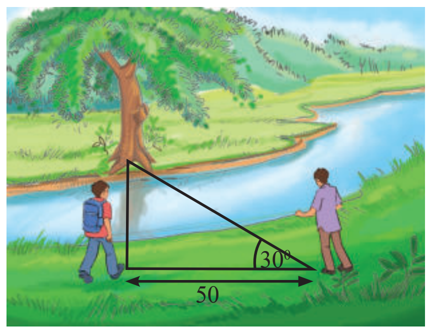
$$\Longrightarrow \frac{\sqrt{3}}{2} = \frac{10}{h - x}$$

$$\Longrightarrow h - x = \frac{20}{\sqrt{3}}$$

$$\Longrightarrow h = x + \frac{20}{\sqrt{3}} = \frac{10}{\sqrt{3}} + \frac{20}{\sqrt{3}} = \frac{30}{\sqrt{3}} = 17.32 \text{ m}$$

So, the length of the pole is 17.32 m (Approx.)

Pair work



You are standing on the bank of a river and looking at a tree on the other side of the river. Move 50 m in such a way that the intersection of your current position with that tree makes an angle of 30°. What is the distance of your previous position from the tree across the river?

Group project

According to the instruction of your teacher, divide into several groups and measure the height of the highest building of your school by using the knowledge of trigonometric ratio. After measuring the height, present the whole process on a poster paper.

Exercise

1. If $\cos\theta = \dfrac{3}{4}$, find out the other ratios of the angle θ.

2. If $12 \cot\theta = 7$, what is the value of $\cos\theta$ and $\csc\theta$?

3. In a right angled triangle ΔABC, $\angle B = 90°$, $AC = 12$ cm, $BC = 13$ cm and $\angle BAC = \theta$. Find out the value of $\sin\theta$,$\sec\theta$ and $\tan\theta$.

4. If $\theta = 30°$, prove that (i) $\cos2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$, (ii) $\tan2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$.

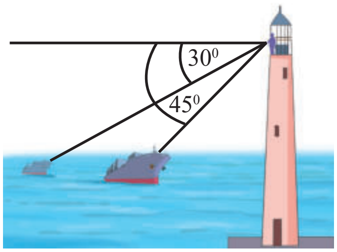
5. If the elevation angle of the top point of the tree is $60°$ at a point on the ground 15 m from its base, find the height of the tree.

6. A ladder with a length of 6 m creates an angle of $60°$ with the base. What is the height of the roof?

7. The elevation angle of the top point of a tower is $60°$ from a point on the ground. The elevation angle of the tower will be $45°$ if is 20 m behind from the previous place. What is the height of the tower?

8. A man standing on the bank of the river observes that on the other side of the river, the elevation angle of the top point of a 100 m tower is $45°$. The man starts journey by boat to the tower. But due to water current, the boat reaches the river side with 10 m. distance from the tower. Determine the distance from starting point of the man to the ending point.

9. A man standing on a tower at the bank of the sea observed that a ship was coming towards the port. At that time, the depression angle of the ship was $30°$. A few moments later, the depression angle became $45°$. If the height of the tower was 50 m, how far did the ship cross during that time?



10. The elevation angle is $45°$ at a distance of 10 m from your office building. If the elevation angle is θ at a distance of 20 m, what is the value of $\sin\theta$ and $\cos\theta$?

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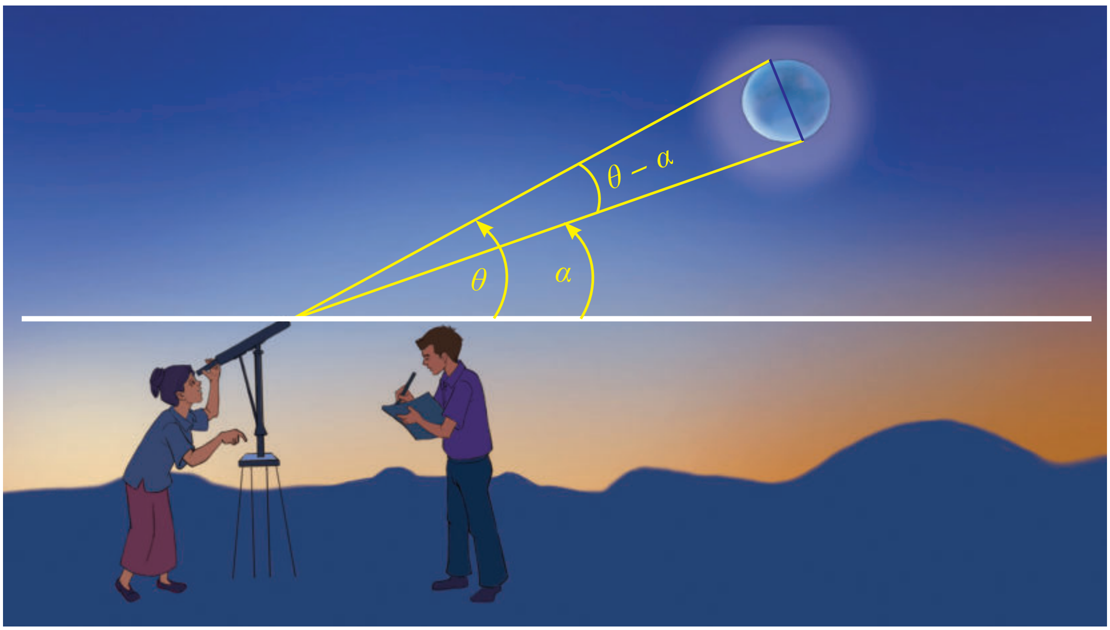
Trigonometry for Angular Distance

You can learn from this experience-

- Concepts, importance, and techniques of measuring trigonometric angles.
- Difference between geometric angles and trigonometric angles.
- Standard positions of trigonometric angles and measures of angles relative to these positions.
- Concept and measurement of coterminal angle, quadrant angle and quadrantal angle.
- Trigonometric ratios in standard position.
- Interrelation of trigonometric ratios of various angles.
- Interrelation of trigonometry and coordinate geometry.
- Radian measurement of angles and the relationship between degrees and radians.



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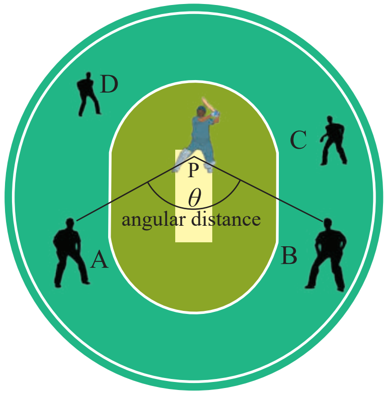
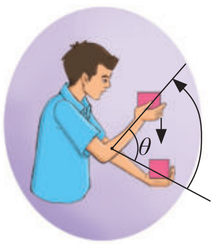
Trigonometry for Angular Distance

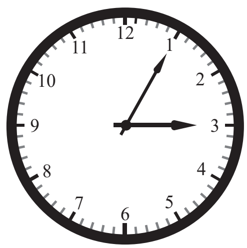


In previous class, we learned to find the linear distance between two objects. Interestingly, apart from linear distance there is another type of distance called angular distance. For example, the adjacent image shows some players on a cricket field. Distances PA and PB directly from batsman P to fielders A and B are called linear distances. But if we want to measure the distance between fielders A and B with respect to the batsman P considered to be at the centre, then that distance is called angular distance. The difference in position of PA to PB with respect to the point P in the adjacent figure is called the angular distance, which is denoted by θ.

Whether we know it or not, we use angular distance in various activities every day. For example, in the game of cricket, a batsman hits the ball keeping in mind the angular distance of the fielders and then runs. Again, we always use different angular distances when working with our hands. The hands of our wall clocks are constantly traversing angular distances. When we measure the distance from one star to another in the night sky, that too is essentially an angular distance. You will find numerous examples where angular distance is used. By measuring angles, we can know the position of many distant objects and determine their size, rotation properties etc. In this chapter we will try to solve such problems by measuring trigonometric angles.



(a)



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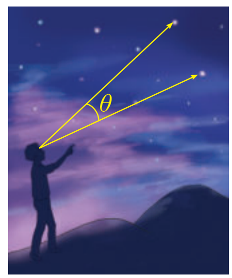

(c)

Work in pairs

Think in pairs and write three examples in the table below where angular distance is used.

To measure angular distance, we use the knowledge of trigonometry. Below this topic is discussed continuously in detail.

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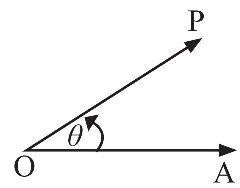
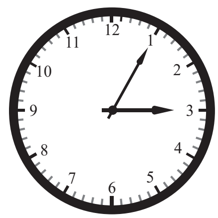
1. Measurement of trigonometric angle



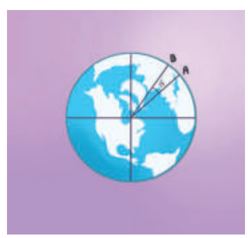
An angle is formed when a ray rotates from an initial position to a terminal position with respect to its starting point. Let OA be an initial ray which turns about the point O and reaches the position OP. So, an angle $\angle AOP$ is formed. Let $\angle AOP = \theta$. Here O is the vertex, OA is the initial line, and OP is the terminal line. The amount by which the ray OP rotates with respect to the point O, keeping the ray OA fixed, is called the angular distance. That is, θ is the angular distance. Angular distance plays a very important role in measurement. Angular distance is usually measured in degrees. Degrees are indicated by the ° symbol. By keeping the ray OA fixed and the ray OP rotated about the point O, angles of different measure are formed. Initially when the ray OP is parallel to the ray OA the angle will be 0°. If the ray OP turns once with respect to the point O and again coincides with the ray OA, then the angle will be 360°. That is, 1° is the angular distance we get dividing one full rotation by 360. 1' (1 minute) is the angular distance we get dividing 1° by 60. That is $1' = \frac{1}{60} \times 1°$. Again dividing 1' by 60 gives the angular distance as $1''$ (1 second). That is $1'' = \frac{1}{60} \times 1'$. Hence, $1'' = \frac{1}{3600} \times 1°$.

Have you noticed that the hands of your home wall clock or table clock or your school's wall clock are always rotating? The hands are repeatedly moving over 12 o'clock. If we imagine a ray starting at 12 o'clock from the center of the clock, what angular distance can you say these hands are traversing? When the hands turn once over 12 o'clock, they traverse an angular distance of 360°. Completing one full rotation once more will cover a distance of 360° + 360° = 720°. Thus, completing each full rotation will add 360° to the angular distance. So, we can see, angle measure can be more than 360° in terms of angular distance. That is, trigonometric angles can be greater than 360°.



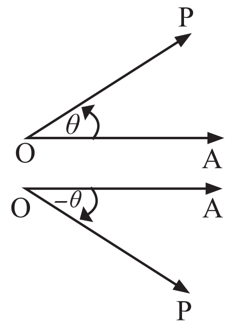
Work in pairs

If the angular distance of two places A and B with respect to the center of the earth in the adjacent figure is 15°, express the angular distance of the two places in seconds. Write the answer in the box below.

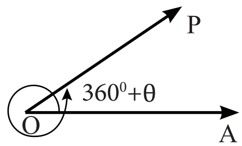


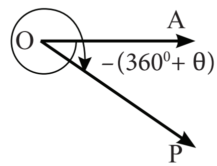
1.1 Positive and Negative Angles

Just as number system has positive and negative numbers, angular distances have positive and negative angles. If the terminal ray OP rotates clockwise with respect to the initial ray OA, then the angle θ is negative, and if the terminal ray OP rotates counterclockwise with respect to the initial ray OA, then the angle θ is positive. Arrows are used to indicate direction of positive and negative angles. Also, negative angles are indicated by a '–' sign before the angle like a numerical expression. Positive and negative angles are indicated in the adjacent figure.



If the terminal ray OP rotates more than 360° counterclockwise or clockwise, then the angle is greater than 360° and we can represent it as in the adjacent figure. Angles greater than 360° are observed in various objects in nature; For example, spiral galaxy, vine arms etc. Can you name more examples that form angles greater than 360°? Think and write in the box below.






Galaxy


creeping tree

Work in pairs

Draw 200° and –230° angles in the blank below using a geometric ruler and protractor.

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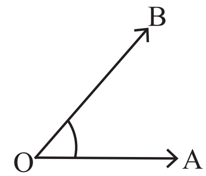
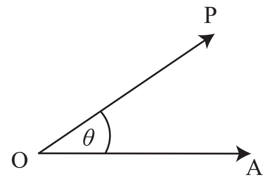
2. Geometric Angles and Trigonometric Angles



From geometry we know that when two different rays meet at a point, an angle is formed at that point. In the figure ∠AOB is a geometric angle. Here the rays OA and OB meet at two points O. Hence angle ∠AOB is formed at point O. ∠AOB is always considered positive in measuring angle. So, the discussion of angles in geometry is limited to 0° to 360° or four right angles.

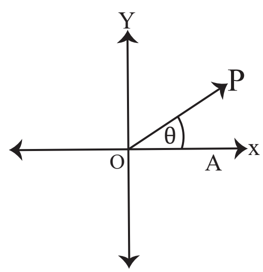


On the other hand, in case of trigonometric angles, keeping the ray OA fixed, rotate OP about the point O to form angles of different measures. Trigonometric angles can be both positive and negative and can be greater than 360°. In the adjacent figure $\theta = \angle AOP$ is a trigonometric angle. This is a positive angle, because the line OP makes an angle θ with the origin line OA in a counterclockwise direction.

Individual task

Draw a geometric angle of 120° in the blank space below. Draw a positive and a negative trigonometric angle of the same measure.

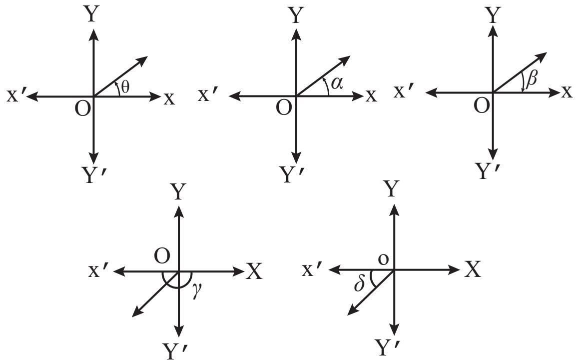
3. Standard position of trigonometric angle



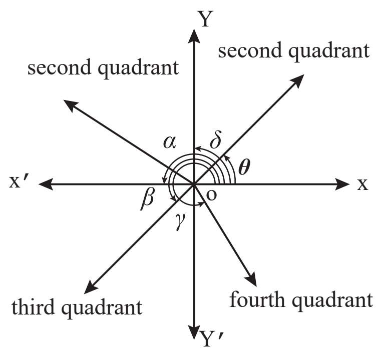
We can represent any trigonometric angle in two-dimensional coordinates or the *xy*-plane. If a trigonometric angle θ is placed in the *xy*-plane in such a way that the vertex of the angle is at O and the initial ray lies on the positive side of the x-axis, then this position is called the standard position of the angle.

Work in pairs

Which of the following angles are in standard positions? Explain the reasons for those which are not in standard position and write in the blank space below.



4. Location of trigonometric angles in different quadrants in standard position



In two-dimensional coordinate geometry the x-axis and y-axis divide the xy-plane into four parts. They are called first quadrant, second quadrant, third quadrant and fourth quadrant. The adjacent figure shows the quadrants. A trigonometric angle in ideal position is located on any of these four quadrants or on an axis. If it is located on a quadrant, it is called a quadrant angle and if it is located on an axis, it is called a quadrantal angle. In the adjacent figure θ is a quadrant angle located in the first quadrant. Similarly, α, β and γ are the quadrant angles which are located in the second, third and fourth quadrants respectively. On the other hand, δ is a quadrantal angle that lies on the positive side of the y-axis.

Example: Find the quadrant of terminal ray OA of angle $\angle XOA = 210°$ in standard position.

Solution: Here $210° = 180° + 30°$.

Since 210° angle is a positive angle, to create this angle the terminal ray OA turns 180° anti-clockwise from the initial ray OX and turns another 30° in the same direction to reach the third quadrant. Hence the terminal ray of the angle is in the third quadrant.

Example: Find the quadrant of terminal ray OA of angle $\angle XOA = -210°$ in standard position.

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Solution: Here $-210° = -180° - 30°$.

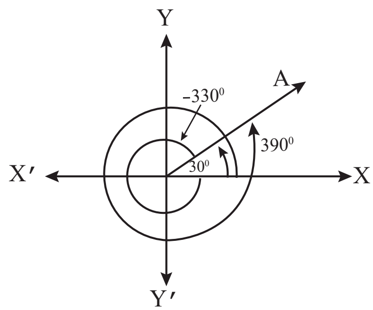
Since $-210°$ angle is a negative angle, to create this angle the terminal ray OA turns $-180°$ clockwise from the initial ray OX and turns another $-30°$ in the same direction to reach the second quadrant. Hence the terminal ray of the angle is in the second quadrant.

Work in pairs

Draw the angles 130°, 400°, $-200°$ and $-750°$ in standard position using ruler and protractor. Determine whether they are quadrant angles or quadrantal angles. State which quadrant the angles are in. Show your work to the teacher.

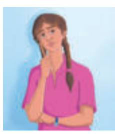
4.1 Coterminal angle

Two trigonometric angles in standard position are called **coterminal angles** if their terminal rays are the same.



Example: 30° and $-330°$ are two coterminal angles. Because, in standard position the terminal rays of these two trigonometric angles are the same. Again the 390° angle is also coterminal with the 30° angle. The adjacent figure shows the angles 30°, $-330°$ and 390°, where OA is the terminal ray.

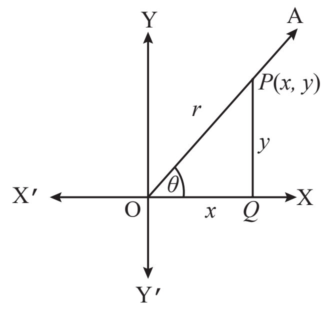
Use some tricks



Find 3 positive and 3 negative coterminal angles of 40° and represent the angles in standard position in the blank below using ruler and protractor.

5. Trigonometric ratios of angles in standard positions

We have learned to find different trigonometric ratios by drawing right-angled triangles in terms of acute angles and seeing that the trigonometric ratio changes as the value of the angle changes. Let us now try to find out the trigonometric ratio using the coordinates by drawing triangles at the standard positions of different angles on the Cartesian plane. You learned in the previous chapter to represent points in two-dimensional coordinates, or the xy-plane. Here we derive the values of various trigonometric ratios from the position of the point on the terminal ray of an angle in standard position.



Let $\theta = \angle XOA$ be a trigonometric angle in standard position whose terminal ray is OA (by the adjacent diagram). Let P(x, y) be any point (except the origin) on OA. So, $OP = r = \sqrt{x^2 + y^2}$.

Then the trigonometric ratios with respect to the angle θ are as follows:

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x},\ \cot\theta = \frac{x}{y},\ \sec\theta = \frac{r}{x},\ \csc\theta = \frac{r}{y}\ .$$

Example: An angle $\theta = \angle XOA$ is in standard position. Determine the trigonometric ratios with respect to the point P(−3, 2) which is on the terminal ray.

Solution: Here $x = -3$, $y = 2$ and $OP = r = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$ Hence the trigonometric ratios are:

$$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{13}} \qquad \cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3}{\sqrt{13}} \qquad \tan\theta = \frac{y}{x} = \frac{2}{-3} = -\frac{2}{3},$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{2} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-3} = -\frac{\sqrt{13}}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{2} = -\frac{3}{2}$$

Individual task:

An angle $\theta = \angle XOA$ is in standard position. Determine the trigonometric ratios with respect to the point P(1, −2) which is on the terminal ray.

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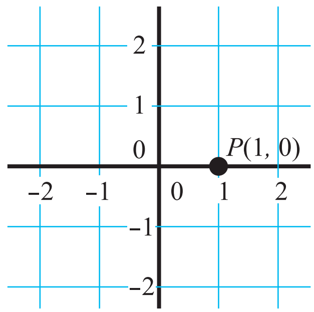
6. Trigonometric ratios of quadrantal angle

In standard position the terminal ray of the quadrantal angle lies on any axis. So, we can find the trigonometric ratio of quadrantal angles with respect to points on different axes.



Example: Find the trigonometric ratios for an angle at the standard position relative to the point P(1, 0) on the positive side of the x-axis.

Solution: Here $x = 1$, $y = 0$ and $OP = r = \sqrt{1^2 + 0^2} = 1$.

So, the trigonometric ratios are:

$\sin\theta = \frac{y}{r} = \frac{0}{1} = 0$ $\quad\cos\theta = \frac{x}{r} = \frac{1}{1} = 1$ $\quad\tan\theta = \frac{y}{x} = \frac{0}{1} = 0$,

$\csc\theta = \frac{r}{y} = \frac{1}{0} =$ undefined $\quad\sec\theta = \frac{r}{x} = \frac{1}{1} = 1$ $\quad\cot\theta = \frac{x}{y} = \frac{1}{0} =$ undefined

Team task/ project

Get divided into groups (minimum of four or multiples of four) as directed by the teacher. Each group will take one graph paper. Draw the x-axis and y-axis on the graph paper and determine the origin O. Each team will take one point on the positive and negative sides of each axis and denote the points by A, B, C, D. The teacher should make sure that the points taken by each group are different. Each team will take one piece of poster paper and glue the graph paper to the top. Now write the answers to the following questions on the poster paper.

- Coordinates of point A on positive side of x-axis:
- Coordinates of point B on positive side of y-axis:
- Coordinates of point C on negative side of x-axis:
- Coordinates of point D on negative side of y-axis:
- Positive angle at terminal ray OA at standard position:
- Positive angle at terminal ray OB at standard position:
- Positive angle at terminal ray OC at standard position:
- Positive angle at terminal ray OD at standard position:

Now each group will find the trigonometric ratio of each angle with respect to the points they have taken and fill in the table below.

Ratio \ $\theta°$	$0°$	$90°$	$180°$	$270°$
sin				
cos				
tan				
sec				
csc				
cot				

Even though different teams have different points, the value of each cell in the above table is the same!!

- Why are the values of each cell in the table above the same? Think logically and reach a decision by discussing with all your team members. Present your decision on poster paper.

Now present your project in front of everyone in one day following the teacher's instructions.

7. Trigonometric ratio of quadrant angles

In standard position the terminal ray of the quadrant angle lies in any quadrant. So we can find the trigonometric ratio of quadrant angles with respect to different quadrant points.

Example: Find the trigonometric ratios for an angle at the standard position relative to the point P(1, 1) on the positive side of the x-axis.

Solution: Here $x = 1$, $y = 1$ and OP= $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Hence the trigonometric ratios are:

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$ $\quad\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$ $\quad\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$

$\csc\theta = \frac{r}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}$ $\quad\sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$ $\quad\cot\theta = \frac{x}{y} = \frac{1}{1} = 1$

Note that the above trigonometric ratios are same as the 45° trigonometric ratios. What is the reason for this? Think and write your reasoning in the blank space below.

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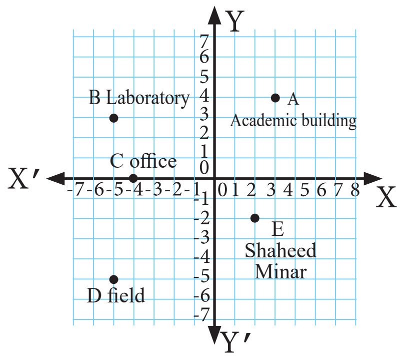
Work in pairs

The adjacent graph paper shows the location of some spots of an educational institution. Find the coordinates of points A, B, C, D, and E. Find the trigonometric ratios for the trigonometric angle θ at the standard position where the terminal ray is the line passing through each of these points from the origin (0, 0) and and complete the table below.



	(x, y)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
A							
B							
C							
D							
E							

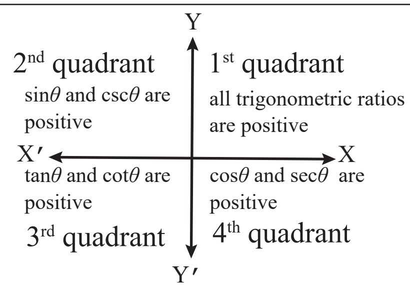
8. Signs of trigonometric ratios in different quadrants

The rotating ray OA of angle $\theta = \angle XOA$ at the standard position in the xy-plane of produces angles of different measure with the initial ray OX depending on the terminal position by counterclockwise rotation. Taking any point P(x, y) on OA will change the coordinates of point P, that is, sign of x and y, due to its position in different quadrants. But if we consider the distance OP = r, it will always be positive. Using this concept, we can find the signs of trigonometric ratios in different quadrants.

Now let's fill in the blanks in the table below. Few are done for you. You have to complete the rest.

Quadrant	Abscissa	Ordinate	Sign of the ratios
First	$x > 0$	$y > 0$	$\sin\theta = \frac{y}{r} > 0$, $\cos\theta = \frac{x}{r} > 0$, $\tan\theta = \frac{y}{x} > 0$ $\csc\theta = \frac{r}{y} > 0$, $\sec\theta =$ $\cot\theta =$
Second	$x < 0$		$\sin\theta =$ $\cos\theta = \frac{x}{r} < 0$, $\tan\theta =$ $\csc\theta =$ $\sec\theta =$ $\cot\theta = \frac{x}{y} < 0$
Third		$y < 0$	$\sin\theta = \frac{y}{r} < 0$, $\cos\theta =$ $\tan\theta =$ $\csc\theta =$ $\sec\theta =$ $\cot\theta =$
Fourth	$x > 0$		$\sin\theta = \frac{y}{r} < 0$, $\cos\theta =$ $\tan\theta =$ $\csc\theta =$ $\sec\theta = \frac{r}{x} > 0$ $\cot\theta =$

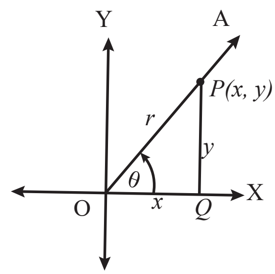
We have seen that the ratios can be positive or negative considering the quadrants. For easy remembering we can use the adjacent diagram. This diagram indicates which quadrants have which ratios positive.



9. Interrelation of Trigonometric Ratios by Difference of Angles

From standard positions in different quadrants of coordinate plane we can determine the interrelationship of trigonometric ratios.

The trigonometric ratios for right triangle ΔOPQ as shown in the adjacent figure are,



$\sin\theta = \frac{y}{r}$, $\quad\cos\theta = \frac{x}{r}$, $\quad\tan\theta = \frac{y}{x}$,

$\csc\theta = \frac{r}{y}$, $\quad\sec\theta = \frac{r}{x}$, $\quad\cot\theta = \frac{x}{y}$

Using all these trigonometric ratios we can determine the interrelationship of different angles.

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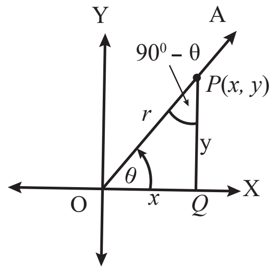
9.1 Using supplementary angles

You know, if the sum of two angles of a triangle is a right angle, one of the two angles is called the complementary angle of the other. Here, $\angle OPQ$ is a complementary angle of θ. That is, $\angle OPQ = 90° - \theta$.



From the angle θ in standard position with respect to the point P(x, y),

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},$$

$$\csc\theta = \frac{r}{y}, \qquad \sec\theta = \frac{r}{x}, \qquad \cot\theta = \frac{x}{y}$$

Again, by the adjacent figure for the right-angled triangle ΔOPQ , for angle $\angle OPQ = 90° - \theta$, base is y and perpendicular is x. Thus, the trigonometric ratios are,

$$\sin(90°-\theta) = \frac{x}{r}, \qquad \cos(90°-\theta) = \frac{y}{r}, \qquad \tan(90°-\theta) = \frac{x}{y}$$

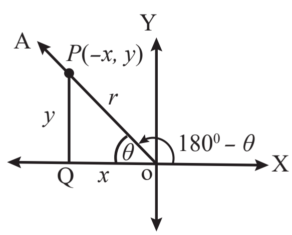
$$\csc(90°-\theta) = \frac{r}{x}, \qquad \sec(90°-\theta) = \frac{r}{y}, \qquad \cot(90°-\theta) = \frac{y}{x}$$

Now fill the table below by observing the above relationships. Two are done for you.

$\sin(90° - \theta) = \frac{x}{r} = \cos\theta$	$\csc(90° - \theta) =$
$\cos(90° - \theta) =$	$\sec(90° - \theta) = \frac{r}{y} = \csc\theta$
$\tan(90° - \theta) =$	$\cot(90° - \theta) =$

9.2 Standard position of the angle is in second quadrant

Now notice at the next diagram. Ray OA turns anti-clockwise and lies in the second quadrant making an angle θ with the negative direction of the x-axis. So the trigonometric angle of terminal ray OA at standard position is $(180° - \theta)$. Take a point P on the ray OA. Draw PQ perpendicular to x-axis from P. Let OQ = x, PQ = y and OP = r. Then the coordinates of point P are $(-x, y)$.



Then, the trigonometric ratios for right triangle ΔOPQ are:

1) $\begin{cases} \sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x} \\ \csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y} \end{cases}$

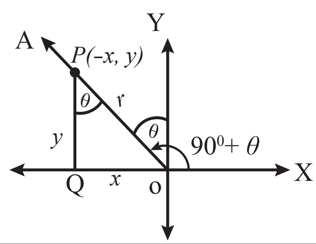
Trigonometric ratios at the standard position of the angle (180° – θ) with respect to the point P(–x, y) are:

2) $\begin{cases} \sin(180° - \theta) = \frac{y}{r}, \quad \cos(180° - \theta) = \frac{-x}{r}, \quad \tan(180° - \theta) = \frac{y}{-x} \\ \csc(180° - \theta) = \frac{r}{y}, \quad \sec(180° - \theta) = \frac{r}{-x}, \quad \cot(180° - \theta) = \frac{-x}{y} \end{cases}$

Using (1) and (2), write the relationship of trigonometric angles to trigonometric ratios in the table below.

$\sin(180° - \theta) =$	$\csc(180° - \theta) =$
$\cos(180° - \theta) =$	$\sec(180° - \theta) =$
$\tan(180° - \theta) =$	$\cot(180° - \theta) =$

Now write down the relation between the trigonometric angles 90° + θ and θ using the diagram:



$\sin(90° + \theta) =$	$\csc(90° + \theta) =$
$\cos(90° + \theta) =$	$\sec(90° + \theta) =$
$\tan(90° + \theta) =$	$\cot(90° + \theta) =$

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Using the above relations we can easily find the trigonometric ratios of some angles in the second quadrant.

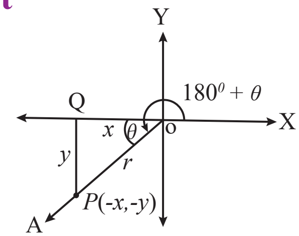
Example: $\cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

Individual task:

Find the values of $\sin120°$ and $\tan135°$.

9.3 Standard position of angles is in third quadrant

Prove the following relations by observing the diagram following the above method.



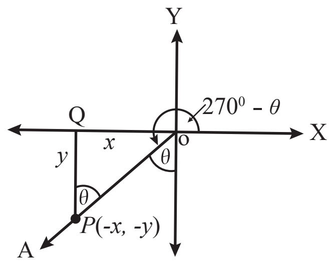
Work in pairs

1. Prove the following relations by observing the diagram.

$\sin(180° + \theta) = -\sin\theta$	$\csc(180° + \theta) = -\csc\theta$
$\cos(180° + \theta) = -\cos\theta$	$\sec(180° + \theta) = -\sec\theta$
$\tan(180° + \theta) = \tan\theta$	$\cot(180° + \theta) = \cot\theta$

2. Prove the following relations by observing the diagram.

$\sin(270° - \theta) = -\cos\theta$	$\csc(270° - \theta) = -\sec\theta$
$\cos(270° - \theta) = -\sin\theta$	$\sec(270° - \theta) = -\csc\theta$
$\tan(270° - \theta) = \cot\theta$	$\cot(270° - \theta) = \tan\theta$

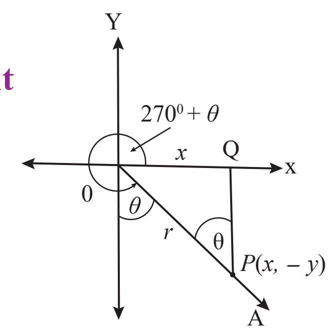


Using the formulas, we can easily find the trigonometric ratios of some angles of the third quadrant.

Example: $\cos225° = \cos(180° + 45°) = -\cos45° = -\dfrac{1}{\sqrt{2}}$

Individual task: Find the value: $\sin210°$, $\tan240°$

9.4 Standard position of angles is in fourth quadrant



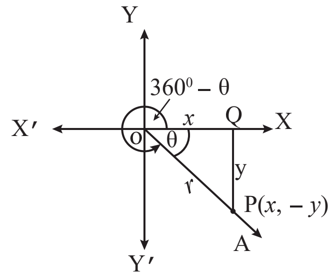
Work in pairs

1. Prove the following relations by observing the figure on the side from the experience of determining the relations of the trigonometric angles of the first, second and third quadrants.

$\sin(270° + \theta) = -\cos\theta$	$\csc(270° + \theta) = -\sec\theta$
$\cos(270° + \theta) = \sin\theta$	$\sec(270° + \theta) = \csc\theta$
$\tan(270° + \theta) = -\cot\theta$	$\cot(270° + \theta) = -\tan\theta$

2. Prove the following relations between trigonometric angles (360°-θ) and θ by observing the figure on the side.

$\sin(360° - \theta) =$	$\csc(360° - \theta) =$
$\cos(360° - \theta) =$	$\sec(360° - \theta) =$
$\tan(360° - \theta) =$	$\cot(360° - \theta) =$

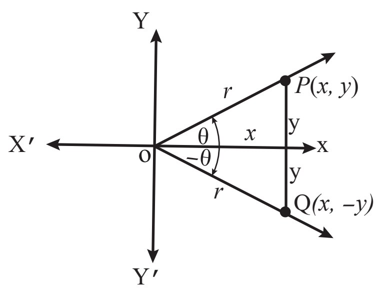


Individual task:

Find the value: sin330°, cos300°, tan315°

3. Prove the following relations between trigonometric angles ($-\theta$) and θ by observing the figure on the side.

$\sin(-\theta) = \dfrac{-y}{r} = -\sin\theta$	$\csc(-\theta) =$
$\cos(-\theta) =$	$\sec(-\theta) =$
$\tan(-\theta) =$	$\cot(-\theta) =$

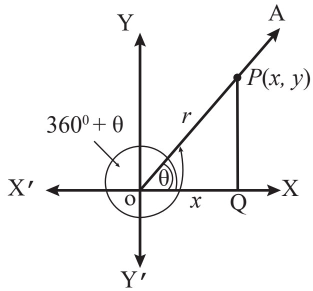


So we can remember the relationship between the trigonometric angles with respect to the axis from the table below.

Angle \ Ratio	sin	cos	tan	csc	sec	cot
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\csc\theta$	$\sec\theta$	$-\cot\theta$
$90° - \theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$	$\tan\theta$
$90° + \theta$	$\cos\theta$	$-\sin\theta$	$\cot\theta$	$\sec\theta$	$-\csc\theta$	$\tan\theta$

$180° - \theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$	$\csc\theta$	$-\sec\theta$	$-\cot\theta$
$180° + \theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$-\csc\theta$	$-\sec\theta$	$\cot\theta$
$270° - \theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$	$-\sec\theta$	$-\csc\theta$	$\tan\theta$
$270° + \theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$	$-\sec\theta$	$\csc\theta$	$-\tan\theta$
$360° - \theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\csc\theta$	$\sec\theta$	$-\cot\theta$

Again if the value of the angle is greater than 360°, we can write it as $(360° + \theta)$. The position of the terminal ray will be same for angle θ and for angle $(360° + \theta)$. As a result, the trigonometric ratios will be the same for both angles. That is



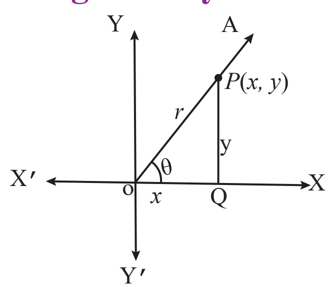
$\sin(360° + \theta) = \sin\theta$, $\quad \cos(360° + \theta) = \cos\theta$ etc.

Example: $\sin 420° = \sin(360°+60°) = \sin 60° = \frac{\sqrt{3}}{2}$

Individual task: Find the value: $\cos 405°$, $\sin 570°$

10. Interrelationship of trigonometry and coordinate geometry

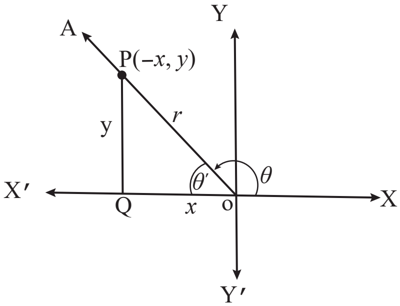
Let P (other than the origin) be a point on the terminal ray OA of positive trigonometric angle $\theta = \angle XOA$ in the xy-plane. Then we can specify the point P in two ways. One is through coordinate geometry and the other is through trigonometric angles. Let PQ be perpendicular to OX through point P. Let $OQ = x$ and $PQ = y$ then in coordinate geometry the coordinates of point P are (x, y). Again if $OP = r$ the point P can be expressed by (r, θ). That is, point P has two forms. One is through the x-axis and y-axis and the other is through the angular distance θ and the distance OP. Here we will establish the relationship between $P(x, y)$ and $P(r, \theta)$. Note here that θ is the trigonometric angle at standard position. Find the value of θ according to the quadrant in which the point $P(x, y)$ lies. To find the value of θ, we need to know the reference angle of θ.

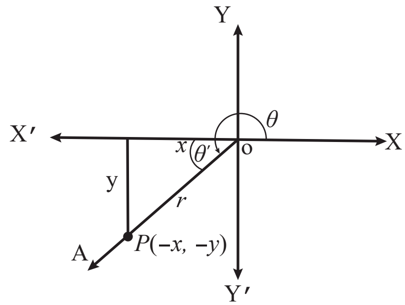


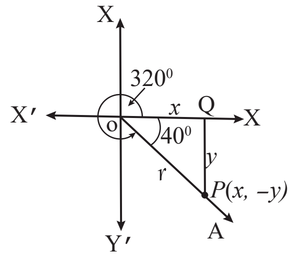
10.1 Reference angle

Let $\theta = \angle XOA$ be a trigonometric angle in standard position. The acute angle OA makes with the x-axis is called the reference angle θ. The reference angle of θ is denoted by θ'.

Below are the reference angles θ' of the angles θ in the second and third quadrants for points P.







Example: Find the value of the reference angle of 320°.

Solution: 320° = 270° + 50°. Hence the angle 320° is in fourth quadrant (see the adjacent figure). The terminal ray creates an acute angle 360°-320° = 40° with the x-axis. So the reference angle of 320° is 40°.

Work in pairs

Find the reference angles of 30°,150°, 280°, 300°, 400° and –240°.

10.2 Expressing P(x, y) in terms of P(r, θ).

Can you tell what is the relationship of x and y with r and θ? Remember, using the Pythagorean theorem we can write,

$$r = \sqrt{x^2 + y^2}$$

So, the relationship of x and y with r and θ is like this:

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta' = \frac{y}{x} \qquad \dots (1)$$

Here, consider the positive values of x and y. From here, after finding the reference angle θ' according to the position of the point P(x, y), find θ.

Example-1

Express P($-\sqrt{3}$, 1) in terms of (r, θ).

Solution: Here $x = -\sqrt{3}$, and $y = 1$. Hence the point P($-\sqrt{3}$, 1) is in the second quadrant.

Hence, $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ and $\tan\theta' = \frac{y}{x} = \frac{1}{\sqrt{3}} = \tan 30°$.

So, θ' = 30°. Therefore θ = 180° – 30° = 150°.

So, in terms of (r, θ), the coordinate of $P(-\sqrt{3}, 1)$ is $P(\sqrt{5}, 150°)$.

On the other hand, if the coordinates of point P are given by r and θ, we can express the coordinates of point P in terms of (x, y) from the following relation.

$$\boxed{x = r\cos\theta \text{ and } y = r\sin\theta}$$

Example-2

Express P(5, 240°) in terms of $P(x, y)$.

Solution: Given that, $r = 5$, and $\theta = 240°$. Therefore,

$$x = r\cos\theta = 5\cos240° = 5\cos(180° + 60°) = 5(-\cos60°) = -5\times\frac{1}{2} = -\frac{5}{2} \text{ and}$$

$$y = 5\sin240° = 5\sin(180° + 60°) = 5(-\sin60°) = -5\times\frac{\sqrt{3}}{2} = -\frac{5\sqrt{3}}{2}$$

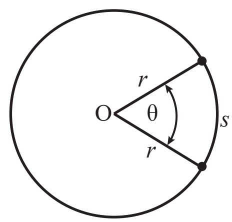
Hence $P(x, y) = P\left(-\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$

Work in pairs

Express $P(\sqrt{2}, 150°)$ in terms of $P(x, y)$.

11. The radian measure of a trigonometric angle

So far we have used degrees as the unit for measuring trigonometric angles. Another important unit of trigonometric angle measurement for solving mathematical problems is the radian. This method uses real numbers to measure trigonometric angles. The angle made at the center by an arc equal to the radius of a circle is called a radian.



11.1 Relation of Angles to Arcs

If the angle made at the center of a circle of radius r by an arc s is θ, then $\theta = \frac{s}{r}$ that is, $s = r\theta$

11.2 Relation between degrees and radians

We know that if a point completes one full rotation on the circumference of a circle of radius r, the trigonometric angular distance will be 360° and the distance traversed by that point will be equal to the circumference, that is, $2\pi r$. Hence, from the relation of arc to angle we get,

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$$360° = \frac{2\pi r}{r} = 2\pi \text{ radian. Therefore,}$$

$$\boxed{1° = \frac{\pi}{180} \text{ radian and } 1 \text{ radian} = \frac{180°}{\pi}}$$

Work in pairs:

1. Express the angles 30°,45° and 60° in radian.
2. Express the angles $\frac{5\pi}{6}$ radian and 20 radian in degrees.

Problem-01: Radius of the Earth is 6440 km. If the positions of Pabna and Sylhet make an angle of 2.5° at the center of the Earth, what is the distance from Pabna to Sylhet? [$\pi = 3.1416$]

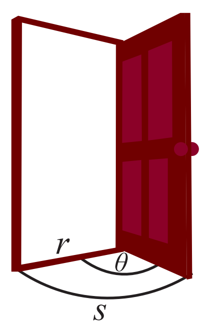
Solution: Here, radius of the Earth, $r = 6440$ km.

Angle created at the center of the Earth by the positions of Pabna and Sylhet, $\theta = 2.5° = \frac{2.5\pi}{180}$ radians.

So, the distance between Pabna and Sylhet is,

$$s = r\theta = 6440 \times \frac{2.5\pi}{180} = 6440 \times \frac{2.5 \times 3.1416}{180} = 281 \text{ km. (approx.)}$$

Problem-02: Suppose the width of your classroom door is 107 cm. The door can't be fully opened because of accommodate one more bench in the class room, but a table is needed to insert into the classroom. If the door of the room is opened 1.4 meter along the perimeter to allow the table to enter the class room, what is the angular distance between the door frame and the door jamb?



Solution: If we open the door, it will create a circular arc on the floor by the edge of it. Let arc length s, radius r and angle θ made by the width of the floor and door jamb on the floor.

Then calculate the length and radius of that arc and fill up the table below.

Length of arc s (meter)	Radius of arc r (meter)

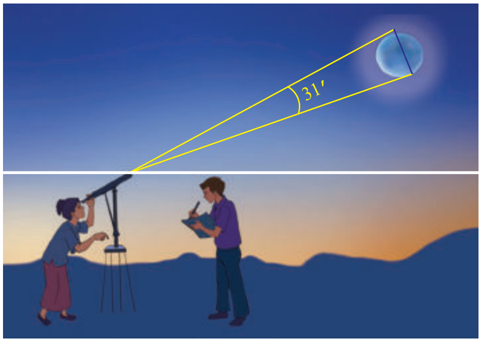
We know, $s = r\theta$

0r, $\theta = \frac{s}{r} = \frac{1.40}{1.07} = \frac{1.40}{1.07} \times \frac{180°}{\pi}$ [$\because$ 1 radian $= \frac{180°}{\pi}$]

$\therefore \theta = \frac{252°}{1.07 \times 3.1416} = \frac{252°}{3.3615} \approx 75°$

So, angular distance 75°

Problem-03: Form a location on the Earth the distance of the Moon is 384,400 km and the diameter of the Moon creates an angle of $31'$ in that position. What is the diameter of the Moon? [$\pi = 3.1416$]



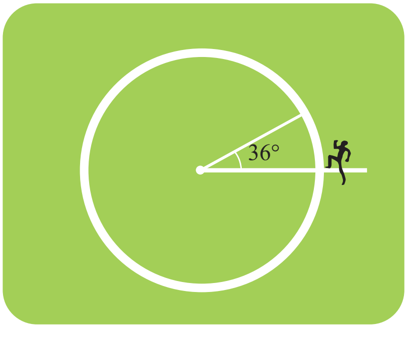
Solution: Here, the distance of the Moon from that location of the Earth, r = 384,400 km and the angle created at that position by the diameter of the Moon is, $\theta = \mathbf{31' = \left(\frac{31}{60}\right)^\circ = (0.517)^\circ}$

Hence, the diameter of the Moon is, $s = r\theta = 384400 \times \frac{(0.517) \times \pi}{180}$

$= 384400 \times \frac{(0.517) \times 3.1416}{180}$ =3468.58 km. (approx.)

Pair work

Suppose your school has 10 m. race in its annual function. So there are made a circular circle in the field. The distance what Wasabi covers in 9 seconds in that race makes an angle of 36° at the center(see side diagram).



a) Find the diameter of the circular circle.

b) How far does Wasabi travel in 5 seconds?

c) What is the angle subtended at the center by the distance he covers in 12 seconds?

d) Determine the speed of Wasabi.

e) The distance covered by Piren in 13 seconds in the same race subtends an angle of 48° at the center. Whose running speed is faster between Wasabi and Piren?

Exercise

1. How many seconds equal $5°$?

2. Construct the angles $30°$, $360°$, $380°$, $-20°$ and $-420°$ using geometric ruler and protractor.

3. Construct the angles $60°$, $90°$, $180°$, $200°$, $280°$, $750°$, $-45°$, $-400°$ at their standard position using geometric ruler and protractor. Determine whether these are quadrant angles or quadrantal angles. State which quadrant the angles are in.

4. Find the values: $\cos 135°$, $\cot 120°$, $\tan 390°$, $\sin(-30°)$, $\sec 300°$, $\csc(-570°)$

5. Find the trigonometric ratios of the angles at standard position where the points A(2, 3), B(–3, 1), C(–4, –4), D(1, –2), and E(–2, 0) are on the terminal rays.

6. Express the following points using r and $\tan\theta$.

   a. A(3, –2) b. B(–2, –1) c. C(–4, 0)

7. Express in radian:

   a. $75°30'$ b. $45°44'43''$ c. $60°30'15''$

8. Express in degrees:

   a. $\frac{4\pi}{25}$ radian b. 1.3177 radian c. 0.9759 radian

9. Radius of the Earth is 6440 km. If the positions of Teknaf and Tetulia make an angle of $10°6'3''$ at the center of the Earth, what is the distance from Teknaf to Tetulia?

10. Radius of the Earth is 6440 km. Suppose, two satellites are positioned above the earth in such a way that they make an angle of $33''$ with the center of the Earth. What is the distance between the two satellites?

Measuring Regular and composite solids

You can learn from this experience-

- Methods of measuring Circumference and Diameter of a Circle
- Understanding Angles
- Methods for measuring Angle's Vertex, Area of Curved Surface, and Volume of Cone
- Methods for measuring Area and Volume of Spheres
- Concept, area, volume of prism and Methods of measuring area
- Methods of measuring area and volume of pyramids
- Concept of formulas for measuring Regular and composite solids and formulaion

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Measuring Regular and composite solids

In the previous class you have learnt about two-dimensional and three-dimensional objects. Two-dimensional objects exist in two dimensions, The two dimensions are length and width. If a triangle, quadrilateral, pentagon or any polygon is drawn on a plane, they lie on the two-dimensional plane. We can only measure their length and width. But when another dimension 'height' is added to these shapes, three-dimensional objects are formed. For example-

Figure-1: two dimensional objects

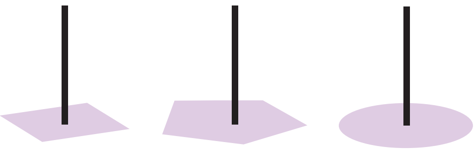


Figure-2: two dimensional objects and height

Almost all the objects we see in nature are three-dimensional objects, that is, they exist in three dimensions, ie, length, width and height. Examples of three-dimensional objects are people, other animals, plants, houses, tall buildings, mountains, etc. Three-dimensional objects are called solids. Every day we have to work with different types of objects. For example, the educational institution you are studing is a solid object, the chair or bench you sit in the class is solid object, even the books, notebooks and pens you use are also solid objects. It is necessary to determine the exact measurements of various necessary solids of daily life in order to make them in the right shape. In this experience we will learn about different measurement techniques of regular solids.

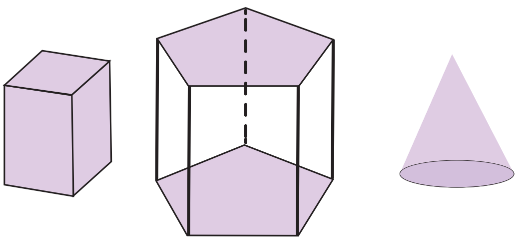


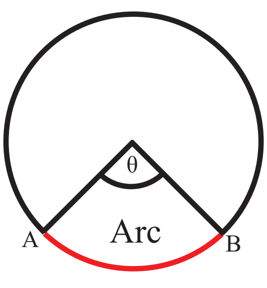
Figure-1: three dimensional objects

By observing the above pictures, it can be seen that three dimensional objects are formed from two dimensional objects. Some solids, that is, cone are formed by sectors. So let us first know about arc and sector.

Measurement of Arc and Sector

You have previously learned about circles in your earlier classes. Additionally, you have also learned how to create circles by cutting paper. You have learned the methods of measuring the circumference and area of a circle. Now, let's delve a bit further into the topic of circles.

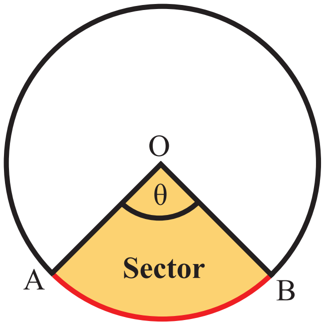
On the circumference of a circle, take any two points A and B. This segment AB is an arc/ circumference of the circle.



Therefore, any part of the circumference of a circle is an arc. Let's consider that the radius of the circle is r units, and the angle formed at the center by the arc AB is $\theta°$.

Furthermore, the measure of an angle formed at the center of a circle is 360°. Arc length and the angle at the center of a circle are proportional to each other. That is,

$$\frac{\text{Arc Length}}{\text{Circumference of Circle}} = \frac{\theta°}{360°}$$



$$\frac{\text{Arc Length}}{2\pi r} = \frac{\theta}{360}$$

$$\therefore \text{ Arc Length} = \frac{\theta}{360} \times 2\pi r \text{ unit}$$

Sector: : An area formed by two radii of a circle and an arc is called a **sector**. Let's consider that the radius of the circle is r units, and the sector AOB is formed at the center with an angle $\theta°$.

Furthermore, the measure of an angle formed at the center of a circle is 360°.

The area of the circular segment and the angle formed at the center are proportional to each other.

$$\frac{\text{Circular Segment Area}}{\text{Circle Area}} = \frac{\theta°}{360°}$$

$$\frac{\text{Circular Segment Area}}{\pi r^2} = \frac{\theta}{360}$$

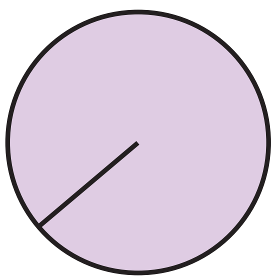
$$\therefore \text{ Circular Segment Area} = \frac{\theta}{360} \times \pi r^2 \text{ square unit}$$

Cone

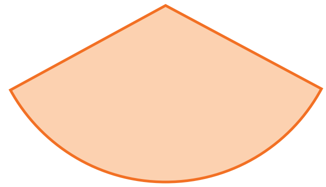
Today, let's engage in another interesting activity involving circles.

Individual Task

Each of you should cut a piece of paper to create a circle. Afterwards, indicate the radius of your respective circles and measure the radius using a tape or a ruler. In the eighth grade, you've learned how to use the radius of a circle to calculate its circumference and area. Therefore, measure the circumference and area of your circle. Write down the measurements in a diagram beside.



Radius	Circumference	Area



Now, based on your circle, if you wish, cut out a circular segment as shown in the diagram. Measure arc length of sector. Use your measurement data to complete the following table.

Arc Length	Circular Segment Area

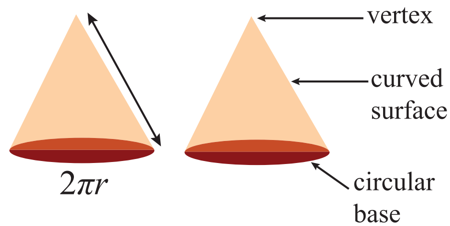
Now, keeping the center of the circular segment exactly aligned with the center of the circle, fold the lower part of the paper as shown in the diagram to create a cone-like structure which is used for eating jhalmuri (spicy puffed rice) . Wow! With this fantastic technique, you've just created an everyday object. Can you guess what kind of object this is? It's a three-dimensional object. This means that by cutting a piece of paper from a two-dimensional circle and folding it, you've crafted a three-dimensional object.

Can you tell the name of this three-dimensional solid? See if the shape reminds you of something like ice cream. Maybe some of you have enjoyed eating this type of ice cream. Can you guess the name of that ice cream? You can call that ice cream a "cone ice cream."

The three-dimensional solid you've created by cutting paper is called a "**cone**" in Bangla. So, wouldn't it be great if we could call the ice cream "cone ice cream" in Bangla as well?

Cone: A cone is a three-dimensional solid formed by taking a circular base and extending a curved surface from a vertex to the edge of the base.

As the diameter of the circle assumes the shape of an angle, the center point of the circle becomes the vertex of the angle. Now, measure the distance from the vertex to the base of the cone, specifically, the marked distance. This distance is known as the slant height of the cone or the height of the slanted surface. Write down the radius of the circular base of the cone and the slant height of the cone together in the diagram below.

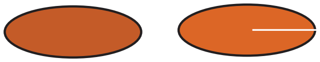


Element Names	Radius of Circular Base	Length of Slant Height of Cone
Measurement		

Is the slant height of the cone equal to the radius of the circular base of the cone, or is it somewhat different? Actually, these two distances are equal to each other.

Notice that the circular segment of the cone is formed by the cone's slanted or curved surface. This region is commonly known as **the curved surface** of the cone or the lateral surface. Therefore, it is evident that the **area of the circular segment and the area of the curved surface of the cone** are equal to each other.

Place the cone that you have created on a piece of paper as shown in the upper diagram and mark its outline using a pencil or pen. Now, if you cut along the marked outline of the cone, what kind of shape can you see? You'll definitely see a circle. Cut the circle neatly. This circle is referred to as the **base of the cone.**



Now measure the radius, circumference and area of this circle using string or scale; Then fill the table below.

Element Name	Radius	Circumference	Area
Measurement			

Now compare the measurement of the circumference with the length of the arc of the circle. Notice that the measurements of the lengths are almost equal or approximately the same. In fact, the measurements are mutually equal. Then, if we examine the measurements that we obtained through the test,

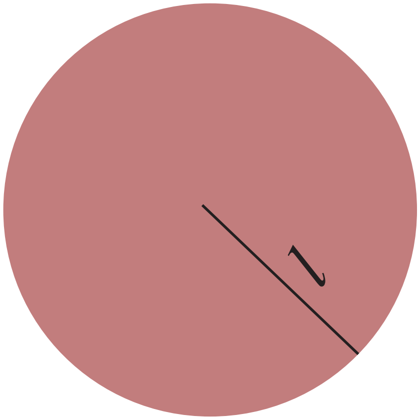
1. Radius of the circle = slanted height of angle
2. Area of the circle = area of curved surface of angle
3. Length of the arc of the circle = Arc length of the angle - Circumference of the circle

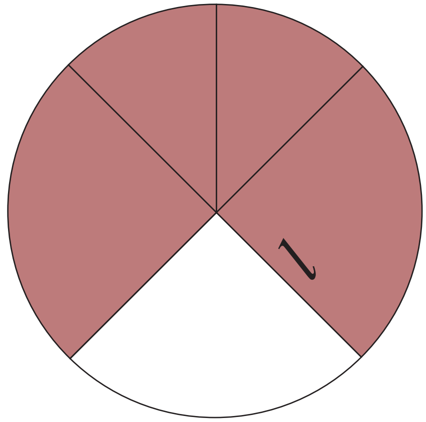
Mathematical Formulation

So far, we have performed measurements using a ruler or scale. However, it's not always feasible to spend so much time on these measurements. Moreover, such measurements may not always guarantee accurate results. For precise measurements, we need mathematical formulas. Let's work on creating mathematical formulas for these calculations.

Individual task

Each of you, cut a piece of paper and create a circle again. Let's assume the radius of your circle is l units. So, the circumference of this circle is $2\pi l$ units and the area is πl^2 square units. Now, divide this circle into four equal parts. Next, cut one of the parts and keep it aside.

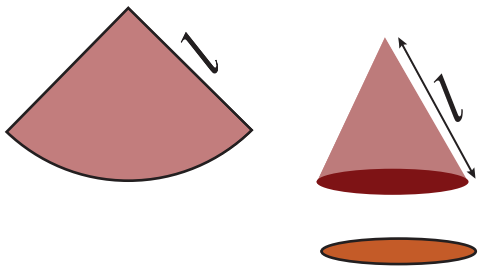




Again, fold the paper to create a cone.

Area of Cone Sector Calculation

Let's calculate the area of the cone sector. Place the angle you've created on a piece of paper. Mark the area covered by the angle's sector by using a pencil or pen. Remember, the radius of the circle related to your angle is *r* unit. Thus, the area of this circle's sector is πr^2 square unit.

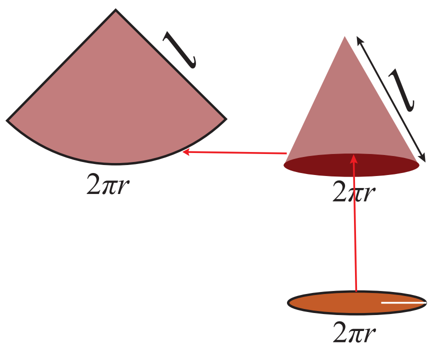


So, the area of the cone sector is πr^2 square units.

Curved surface area of cone

Now, let's determine the area of the curved surface of the cone. The radius of the circle you've drawn is r units. Consequently, the circumference of this circle is $2r\pi$ units.

Do you remember the relationship between the circumferences of the smaller and larger circles? Think about it. When you cut a circle sector from the larger circle to create your cone, the curved length of that sector matches the circumference of the larger circle. This is because the end point of that sector creates the circumference of the smaller circle.



$\therefore$ The circumference of the Inscribed Circle = The Arc Length of the Circumference of the Larger Circle.

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Again, the radius of the larger circle l = the inclined height of the corner l.

Again, since the circumference of this circle is $2\pi r$ units, the length of the arc of the larger circle is also $2\pi r$ units.

Moreover, the area of this inscribed circle is equal to the area of the sector formed by the obtuse angle; because creating cone involves rotating the sector itself. Therefore, if we can calculate the area of this inscribed circle, we can also determine the area of the sector formed by the obtuse angle.

1st Method

We already have the following information: the circumference of the larger circle is $2\pi l$ units and its area is πl^2 square units. so we can write,

If the circumference of the larger circle is $2\pi l$ units, the area = πl^2 square unit

$\therefore$ If circumference of larger circle is 1 unit then area = $\frac{\pi l^2}{2\pi l}$ square unit

$\therefore$ If the circumference of the larger circle is $2\pi r$ unit, then area = $\frac{\pi l^2}{2\pi l} \times 2\pi r$ square unit

$$= \frac{\pi l.l}{l} \times r \text{ square unit}$$

$$= \pi rl \text{ square unit}$$

$\therefore$ For circumference of the larger circle $2\pi r$ units, area of incircle is πrl square units.

Now, area of curved surface of cone = area of circle

$$= \pi rl \text{ square unit}$$

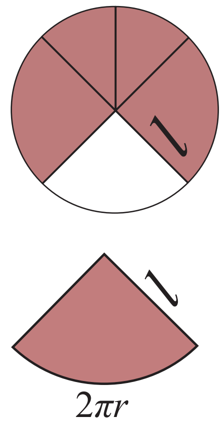
$\therefore$ When the radius of the cone is r units and the slant height is l units, the surface area of the cone = πrl square units.

2nd Method

What you already know is this: the circumference of the larger circle is $2\pi l$ units, and the length of the arc of the circle is $2\pi r$ units.

Now, do you remember, there's an interesting relationship between the circumference of the larger circle and the length of the arc of the circle? In other words, there's a fascinating connection between $2\pi l$ and $2\pi r$?

Actually, the length of the arc of a circle is one-fourth of the circumference of the larger circle. In other words,



∴ One-fourth of the circumference of the larger circle = Length of the arc of the circle

or, $\frac{2\pi l}{4} = 2\pi r$

or, $\frac{l}{4} = r$

∴ $l = 4r$ ... ...(1)

Again, can you say if there is a relationship between the area of the larger circle, πl^2, and the area of the sector of a circle? In other words, what relationship exists between πl^2 and the area of the sector of the circle?

Actually, the area of the sector of a circle = One-fourth of the area of the larger circle.

$$= \frac{\pi l^2}{4} \text{ square units}$$

$$= \frac{\pi l.l}{4} \text{ square units}$$

$$= \frac{\pi l.4r}{4} \text{ square units [Substituting } l = 4r \text{ from equation (1)]}$$

∴ Area of the sector of a circle = πrl square units

Now,

Area of the segment of the cone = Area of the sector of the circle

$= \pi rl$ square units

3rd Method

Let us learn another method of finding the area of a curved surface of a cone.

Step 1: Consider any cone. Imagine that this cone is formed by an arc on a circle with a radius of r units and a length of the inclined plane of l units. Therefore, the circumference of this angle is $2\pi r$ units.

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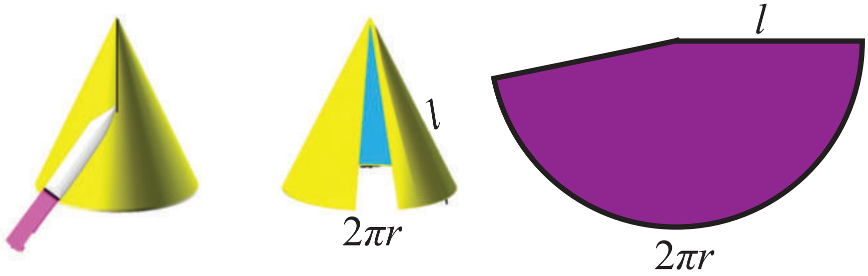


Step 2: Cut the cone carefully following the instruction in the diagram and place it on a piece of paper or on a flat surface. If you observe closely, you'll notice that the curved side of the angle forms a circular segment. The center of this circular segment is the vertex of the angle, the radius of the inclined plane of the angle is l units and the length of the arc is $2\pi r$ units of the perimeter of the angle.

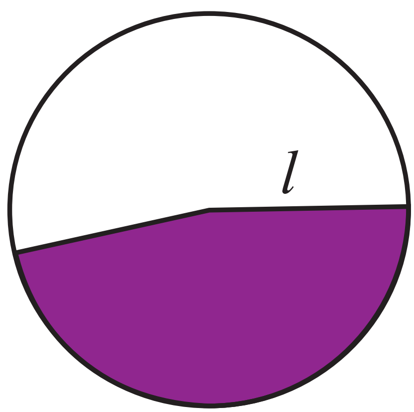
Using a ruler, measure the radius of the circular segment. Then, determine the area of the circular segment and complete the table below

Elements Name	Radius	Area of Circular Segment
Measurement		

The area of this circular segment is, in fact, the area of the sector formed by the angle. However, we aim to establish a mathematical formula.

Step 3: Using the center and radius l let us finish drawing the circle associated with the circular segment. Consequently, the circumference of the constructed circle is $2\pi l$ units, and the area is πl^2 square units.

Step 4: The area of a circular segment is proportional to the area of the corresponding sector of the circle's circumference. Therefore,



$$\frac{\text{Area of Circular Segment}}{\text{Total Circle Area}} = \frac{\text{Arc Length}}{\text{Total Circle Circumference (Circumference)}}$$

or, $$\frac{\text{Area of Circular Segment}}{\pi l^2} = \frac{2\pi r}{2\pi l}$$

Or, Area of Circular Segment $= \dfrac{2\pi r}{2\pi l} \times \pi l^2$

Or, Area of Circular Segment $= \dfrac{r}{l} \times \pi l.l$

Or, Area of Circular Segment $= \pi rl$

$\therefore$ Or, Area of Circular Segment $= \pi rl$ square units

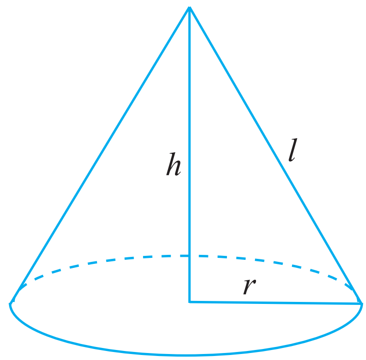
Now, area of curved surface of cone = area of circular segment

∴ Area of curved surface of cone = πrl square unit.

∴ If the radius of the base of the cone is r units and the slant height is l units,

∴ Area of the curved surface of cone = πrl square unit

Height of the cone



The height of the cone is the length of the perpendicular drawn from the apex of the cone to the base.

Notice that when you have a cone with the radius of the base as r, height as h, and the slant height as l, a right triangle is formed. Can you tell what type of triangle is this?

Individual task

Analyzing the triangle beside, complete the 2nd row of the following diagram:

1	Triangle Name Based on Angle	Value of angle created by r and h	r and h are names of sides	name of l side	The relationship between r, h and l sides	The name of the theorem related to triangle
2						

You must have understood, a right triangle is formed by radius r, height h and length l of slanted plane with hypotenuse l. So, according to the Pythagorean theorem,

$$l^2 = r^2+h^2$$

$$\therefore\ l = \sqrt{r^2+h^2}$$

$$\therefore\ l = \sqrt{r^2+h^2}$$

Then, the area of the curved surface of the angle = πrl square unit

$= \pi r\sqrt{r^2+h^2}$ square unit

Surface area of cone

The total surface area of an angle is the sum of the surface area of the angle and the area of the curved surface of the angle.

So,

Total surface area of angle = surface area of angle + area of curved surface of angle

$$= (\pi r^2 + \pi r l) \text{ square unit}$$

$$= \pi r(r + l) \text{ square unit}$$

$$= \pi r\,(r + \sqrt{r^2+h^2}\,) \text{ square unit}$$

another way, Total surface area of angle = $(\pi r^2 + \pi r l)$ square unit

$$= \pi r^2 + \frac{1}{2}(2\pi r)l \text{ square unit}$$

$$= \text{surface area} + \frac{1}{2} \times \text{perimeter of base} \times \text{slant height}$$

> ∴ If the ground radius of the cone is r unit, height, h unit and slant height, l unit,
>
> total surface area of the angle will be, $= \pi r\,(r + l)$ square unit
>
> $$= \pi r\,(r + \sqrt{r^2+h^2}\,) \text{ square unit}$$

Problem 1: On the birthday of one of your friends, you want to give him or her gift a hat made of paper having a height of 35 cm. Measuring the circumference of your friend's head with a thread, you get 48 cm. How much paper do you need to make the hat?

Problem 2: Some of your school friends participated in a science fair together. You want to build a cone shaped tent of 5 metres height with poles in the middle to store all your belongings. If this tent covers 150 square meters of land, how much fabric will be required for the tent? If the price of cloth is 160 taka per square metre, how much will the total cost be?

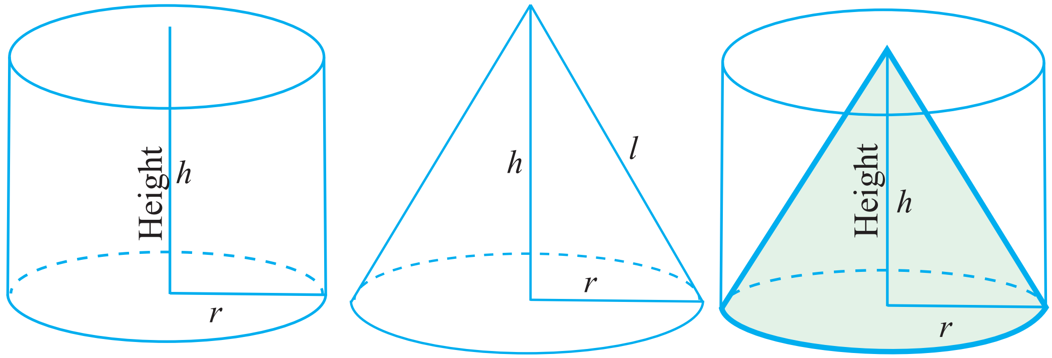
Volume of cone

You have already learned how to create a cone by cutting papers and have studied cylinders in detail in previous classes. Let's try to explore more with hands-on activities.

Group work

Using sturdy paper, cardboard, or plastic board that can be easily bent, create a cylinder and a cone in such a way that the heights of both remain equal or the same. Seal the bottom of the cylinder with cardboard, leaving the top open, and keep the base of the cone open so that something can be kept inside.

Similarly, keep the base of the cone open so that something can be placed inside it as well. Create an arrangement that mirrors the illustration provided.



To find out if your constructed cone and cylinder are made correctly, place the cone inside the cylinder as shown in the last illustration. Check if the base radius and height of both the cone and cylinder are equal or the same. If there are any errors in the construction, make the necessary adjustments. Once everything is in order, your cone and cylinder have been successfully created.

Our wait is now over. Let's proceed to the exciting main part!

First, hold the cone as if you're having spicy puffed rice or an ice cream cone. Then, fill the cone with some rice, flour, or any other materials in a way that the base of the cone becomes flat due to the material inside. Then, place the elements of that cone inside the empty cylinder. You'll notice that some part of the cylinder gets filled.

Similarly, for the second time, fill the empty cone with the same material and place it inside the same cylinder. You'll see that most of the cylinder gets filled.

Now, for the third time, when you fill the empty cone with the same material and place it inside the same cylinder, what do you see? The cylinder becomes completely filled from top to bottom.

Isn't it fun? With the three cone-shaped materials, you've successfully filled the entire cylinder to the brim.

You can also try the above experiment with water or any other liquid substance.

For example, using a readily available water-resistant plastic bag, create a cone and a cylinder in such a way that the cone and the cylinder are positioned to allow water or any liquid substance to be poured in without leaking. Then, following the same steps like before, pour water or any liquid substance into the cone and place it inside the cylinder. You will see that the liquid remains inside the cylinder and cannot escape.

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Likewise, repeat the procedure using the same plastic bag. Fill the empty cone with the same liquid substance and place it inside the same cylinder. You will notice that most of the cylinder gets filled to the brim.

Furthermore, you can also verify inversely that three cone-shaped containers can be filled by a single cylinder of the same substance.

Therefore, the volume of the 3 cone-shaped containers = Volume of 1cylinder

As we have learned in previous classes, when the base radius of a cylinder is denoted as r and its height is denoted as h and volume = $\pi r^2 h$ cubic units.

Therefore, the volume of the 3 cone-shaped containers = $\pi r^2 h$ cubic units.

$\therefore$ Volume of 1 cone = $\frac{1}{3}\pi r^2 h$ cubic units

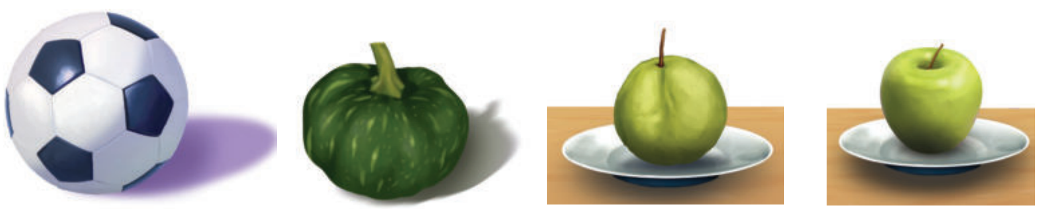
$\therefore$ If the base radius of a cone is denoted as r units and its height is denoted as h units, then:,

Volume of the cone = $\frac{1}{3}\pi r^2 h$ cubic units = $\frac{1}{3}$ × (area of base × height) cubic units.

Problem 3: On your birthday, you've invited six friends to your house for a special home-made ice cream treat. The ice cream cone has a base radius of 6 cm and a height of 15 cm. Along with your friends, both you and your mother also want to enjoy an ice cream cone. Your mother does not know how much ice cream needs to be prepared. Can you figure out the amount of ice cream that needs to be made? Can you measure it in advance and let your mother know?

Sphere

You have learned about spheres in the eighth grade. You have also learned to create a circle by cutting paper. Can you name the objects below? Surely you can name these. Because we use these objects in our everyday life regularly. Among these, one might be a sports equipment, another might be a fruit, and yet another might be a type of vegetable.



Names of Several Types of Balls: Tennis Ball, Cricket Ball, Football, Basketball

Several Round-shaped Fruits: Guava, Mandarin Orange, Apple, Sweet Orange

Several Round-shaped Vegetables/Fruits: Sweet Pumpkin

Round-shaped Toys: Ball

Item Used for Drawing/Writing with a Round Shape: Compass

Write a few more names of items with such a round shape in the following list.

Wood apple				

Can you tell what kind of objects with a round shape we use in our daily life? All of these are three-dimensional solids. Can you also tell what these three-dimensional solids are called in the language of mathematics? Objects of this type are called spheres in mathematical terms.

Sphere: When a circle is rotated about its fixed diameter, the three-dimensional object formed by the rotation is called a sphere. In three-dimensional geometry, a sphere is a perfectly symmetrical solid in the shape of a round ball.

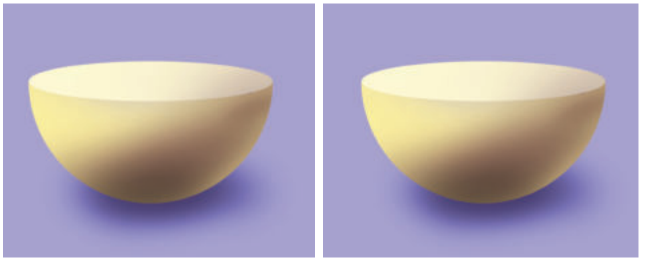
When we consume fruits or vegetables of spherical shape, we often don't need to know the exact dimensions of their skin or peel. However, when we create objects like a soccer ball, we need to be aware of the dimensions of its surface, which includes the skin or related materials. Therefore, knowing the surface area of materials like leather or similar substances becomes crucial when crafting items like soccer balls. In our daily lives, having a good understanding of the surface area of such three-dimensional objects is essential for effective management of our regular life. Let's learn the techniques to determine the surface area of real-life objects and spheres that we commonly use.

Surface area of sphere

Group Work

Step 1: Take a spherical object like a plastic tennis ball.

Step 2: Cut the ball exactly in the middle, creating two hemispheres as shown in the diagram. You will notice that the shape of the two hemispheres corresponds to the shape of a sphere.



Step 3: Take any hemisphere and place its cut side up or down on a piece of paper; Then mark around the hemisphere with a pencil or pen on the notebook. Then lift the hemisphere off the paper.

What do you see? Can you recognize the picture on the paper? Tell me what the picture is. Yes, it is an image of a circle.

Step 4: Carefully cut the circle using scissors. Then measure the radius of the circle. Finally, measure the area of the circle and complete the table below.

Name of elements	**Radius**	**Surface area**
Measurement		

On another piece of paper, cut out another circle of the same measurement of the circle that you have obtained by cutting one. Then cut the two circles conveniently into small pieces and paste them on the hemisphere. You will see, the whole hemisphere is covered by two pieces of paper and no piece of paper is left. Then it is seen that the surface area of the hemisphere is equal to the sum of the areas of the two circles. Therefore, you can conclude that the surface area of the full sphere is the sum of the surface areas of four such circle segments.

Measure the sum of the areas of four circles by looking at the area of a circle from the table; Then fill the table below.

Area of two circles	Area of four circles	Surface area of hemisphere	Area of sphere

However, you can find the area of the sphere practically.

Let us now try to find the mathematical formula for the area of a sphere.

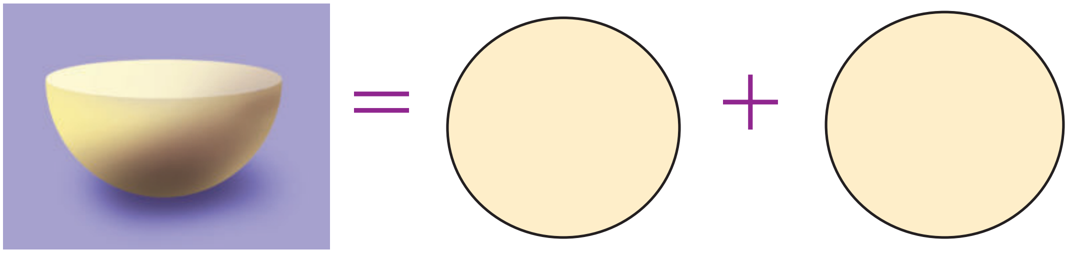
Assume that the circle you cut has radius r units. Then you understand, the radius of the sphere is also r unit.

You learned in the previous class that a circle whose radius is r units has an area of πr^2 square units. Then the area of your inscribed circle is also πr^2 square units.

Now we can say,

Surface area of hemisphere = sum of area of two circles.

It can be shown with the help of the figure below

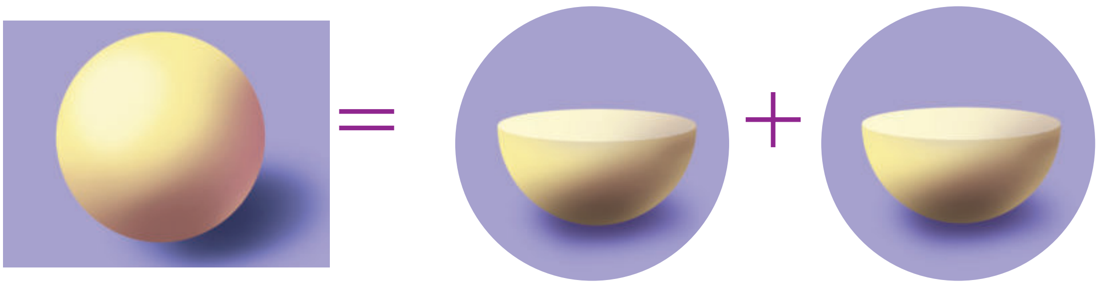


Surface area of hemisphere = $(\pi r^2 + \pi r^2)$ square units

$= 2\pi r^2$ square units

Now, area of a solid sphere = sum of surface areas of two hemispheres

It can be shown in the figure below:



Area of the sphere = $(2\pi r^2 + 2\pi r^2)$ square units

$= 4\pi r^2$ square units

$\therefore$ If the radius of the sphere is r unit, the area of the sphere is = $4\pi r^2$ square unit.

Problem 1: To test how day and night occurs, you want to make a 16 cm diameter globe out of thin plastic paper. If the price of plastic paper is five taka per square cm, how much money will you spend to buy plastic paper to make the globe?

Problem 2: The diameter of an ideal hemisphere is 10 cm. Find the curved surface or surface area, total surface area and volume of the hemisphere. If it costs 2 taka per square centimetre to paint the entire hemisphere, how much money is required in total?

Volume of sphere

When we eat guava or apple in our daily life, we all know that a large guava or apple has more nutrients than a small guava or apple has. But if it is asked how much food ingredients one guava or apple contains exactly, we cannot say exactly. To know the exact amount of ingredients in that guava or apple, it is necessary to determine the volume.

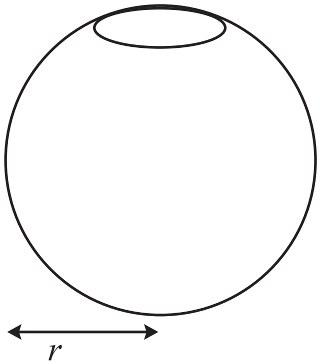
Suppose you want to build a spherical aquarium at home. You want to fill two-thirds of your aquarium with water and one-third with air. You still need to determine the volume of your aquarium.

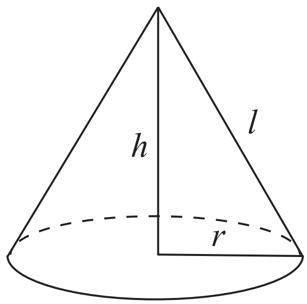
Again, to know how much space should be kept vacant in a football, it is also necessary to know the volume of the football.

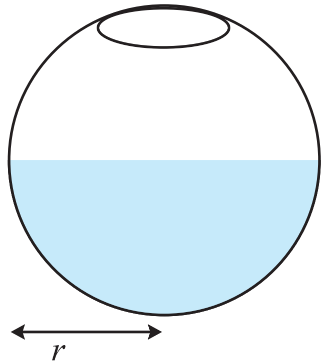
Some of the daily life problems as discussed above are all about spheres and the volume of spheres. So we need to know how to calculate the volume of a sphere. Let us learn the technique of finding the volume of a sphere.

Group work

Collect one plastic toy ball in each group on your own initiative. The ball shall be such that water can be inserted into or withdrawn from it. Then measure the radius of the ball and record it in your notebook. You have already learned how to make cone. Now construct an angle in such a way that the radius of the base of the angle is equal to the radius of the ball. Moreover, care should be taken that the height of the cone is equal to the diameter of the ball. Make the cone Waterproof with readily available polythene or any other material to allow water to pour into it.







Correct any construction errors if there are. Then your cone and balls are ready.

Our wait is over. Now let's get to the main part of the fun.

First make sure that the cone is filled with water. Then fill the cone with water and pour it into the empty ball. If the ball is transparent, you can see that the ball is roughly half full.

Similarly fill the empty cone a second time with water and pour it into the ball.

What do you notice? The ball is full to the brim.

What a fun, right? A ball is filled with two cones of water.

You can also do the above test with juice or any other liquid.

Again, you can check in a reverse way; A ball full of water can fill two such cones with water and no water remains in the ball.

So, we were able to prove practically that the capacity of a sphere is equal to the capacity of two cones if the radius of the base of the cone is equal to the radius of the sphere and the height of the cone is equal to the diameter of the sphere.

So, volume of 1 sphere = volume of 2 cones

As we have learned in previous sessions, if the base radius of a corner is r units and height is h units, volume = $\frac{1}{3}\pi r^2 h$ cubic units.

Therefore, volume of 2 cones = $\frac{2}{3}\pi r^2 h$ cubic units

Volume of 1 sphere = volume of 2 cones

$\therefore$ Volume of 1 sphere = $\frac{2}{3}\pi r^2 h$ cubic units

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or, Volume of 1 sphere $= \frac{2}{3}\pi r^2.2r$ cube unitss [Here $h = 2r$]

$\therefore$ Volume of 1 sphere $= \frac{4}{3}\pi r^3$ cube units

$\therefore$ If the radius of the sphere is unit r unit , the volume of the sphere $= \frac{4}{3}\pi r^3$ cube units

Problem 3: A hollow iron sphere has outer radius 8 cm and thickness of iron 3 cm. What is the volume of the hollow part of the sphere? Moreover, a pure sphere was made from the iron used in that sphere. If it costs Rs 1.75 per square centimeter to paint a solid sphere, how much will be the total cost?

Problem 4: Three solid plastic balls of radii 5 cm, 7 cm and 11 cm are melted to form a new solid ball. Find the surface area of the new ball.

Prism

We use so many things in our daily life. The items below are also not exceptions. Several pieces of furniture, tables, and so on are visible here. The last item, again, is used by many people as a tool. People in the city also use this last item for fish farming or breeding inside their homes as a hobby. Its name is an aquarium. So whatever the case, all of these are individual objects. There are so many other things of this kind that we use every day.

Can you also make a list of a few more items or solid objects of this type? However, the specific characteristic of the objects given above is that each of them has the same top and bottom surfaces. In mathematical terms, they are called congruent. Moreover, the top and bottom surfaces are also parallel to each other. The lateral surfaces are rectangular or parallelogram in shape. In mathematical language, these types of solid objects are called **prisms**.

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Prism: A solid object with two congruent and parallel polygonal bases, and lateral faces that are parallelograms, is called a prism. The two opposite, congruent, and parallel bases of a prism are referred to as the **bases of the prism**. The other faces of the prism, excluding the bases, are called the lateral faces of the prism. When the lateral faces of the prism are rectangular, the prism is called **a right prism or upright prism**. On the other hand, if the lateral faces of the prism are not rectangular but have other shapes, the prism is called **an oblique prism or slant prism**.

A prism is named on the basis of the shape of its base on the ground. For example, if the base of a prism is triangular in shape, it is called a triangular prism. Likewise, if the base of a prism is quadrilateral in shape, it is called a quadrilateral prism. If the base of a prism is pentagonal in shape, it is called a pentagonal prism, and so on.

Can you tell what type of prisms the above objects are? Based on the images above, identify the type of prism each object is according to the shape of its base on the ground and complete the following table.

Object no.	1st object	2nd object	3rd object	4th object
Type of prism	Rectangular Prism			

If the base of a prism is not of polygonal shape, but the lengths of its lateral edges are equal, then it's called a right prism. In other words, when the lateral sides of the prism are equilateral polygons or regular polygons, it is known as a right prism. On the other hand, if the base of the prism is not an equilateral polygon, it i's called an oblique prism. Another name for a right prism is a regular prism. In fact, right prism is more commonly known as regular prism. Irregular prism refers to a prism where the base is not an equilateral polygon.

The distance between the center of a prism's base is called the **height of the prism**.

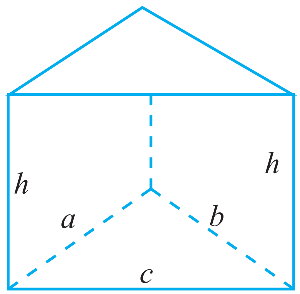
So, when analyzing the characteristics of a prism, it can be observed that rectangular solids and cubes are each a type of prism. Can you now identify what type of prism a rectangular solid and a rectangular cube are? Combine your concepts about prisms and complete the table by putting tick mark ☑ appropriately.

	Right Prism	Oblique/ Skewed Prism	Regular/ Uniform Prism	Irregular/Non-uniform Prism
Rectangular solids				
Cube				

Surface Area of Prism

Now let us try to calculate the surface area of a prism. A prism has two bases and several lateral faces. The surface area of these bases and lateral faces add up to the total surface area of the prism.

∴ Therefore, the total surface area of the prism = Surface area of two bases + Surface area of lateral faces.



First, let's create the formula for the surface area of a triangular prism. This triangular prism has two bases and three lateral faces. Each of these lateral faces is a rectangle.

Therefore, the total surface area of this prism will be equal to the sum of the area of the two bases and the combined area of the three rectangular lateral faces.

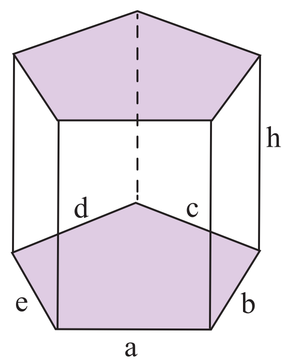
∴ Total surface area of the prism = 2× (area of the base.) + combined area of the three rectangular lateral faces.

= 2× (area of the base) + $(a \times h + b \times h + c \times h)$

= 2× (area of the base) + $(a + b + c) \times h$

= {2× (area of the base) + perimeter of the base× height} square unit

Now let us find out the formula for the surface area of a pentagonal prism. This pentagonal prism has two congruent pentagonal bases and five lateral faces. Each of these lateral faces is a rectangle.



∴ Total surface area of the prism = 2 × (base area) + Sum of the areas of five rectangular lateral faces.

= 2× (area of the base) + $(a \times h + b \times h + c \times h + d \times h + e \times h)$

= 2× (area of the base) + $(a + b + c + d + e) \times h$

= {2× (area of the base) + perimeter of the base × height} square unit

> ∴ Total surface area of the prism = {2× (area of the base) + perimeter of the base × height } square unit

Now, if the base of the prism is a regular polygon, meaning that each side of the base polygon has equal length,

∴ Total surface area of the prism = 2× (area of the base) + $(a + b + c + d + e) \times h$

= 2× (area of the base) + $(a + a + a + a + a) \times h$

= 2× (area of the base) + $(5a \times h)$

= 2× (area of the base) + (Number of sides × Length of each side × Height) square units

So, for a regular prism, if the base is a regular polygon with n sides and each side of the base has a length of a,

∴ Total surface area of the prism = 2× (area of the base) + $(na \times h)$ square units

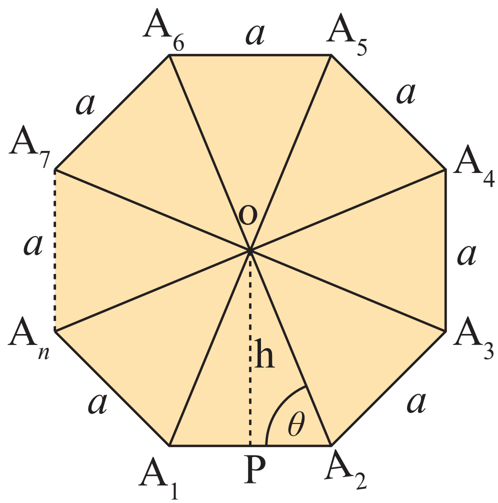
= 2× (area of the base) + (Number of sides × Length of each side × Height) square units

> ∴ So, for a regular prism, if the base is a regular polygon with n sides and each side of the base has a length of a,
>
> ∴ Total surface area of the prism = 2× (area of the base) + $(na \times h)$ square units
>
> = 2× (area of the base) + (Number of sides × Length of each side × Height) square units

Area of a Regular Polygon

We have learned to determine the areas of triangles and quadrilaterals in previous classes. However, we also need to know how to calculate the area of a polygon with more than four sides, such as a pentagon, hexagon, heptagon, octagon, or any polygon with more sides. Let us learn how to calculate the area of any regular polygon.

Think of a convex polygon with an n numbered sides $A_1\ A_2\ A_3\ A_4\ A_5 \ldots \ldots A_n$ where O is the center, and each side has a length of a. When you add the center and the vertices of a convex polygon with 'n' sides, you create 'n' congruent isosceles triangles with equal sides. Therefore, each of these small triangles has two sides that are equal in length. The center of a convex polygon refers to the center of its incircle.



Due to the equal length of all sides in a convex polygon, the angles of the polygon are also equal to each other. Therefore, in a convex polygon with n sides, the number of angles will also be n.

First, let's calculate the area of any triangle, such as ΔOA_1A_2. The height of ΔOA_1A_2 is OP = h and $\angle OA_2A_1 = \theta$

$\therefore \angle A_1A_2A_3 = \angle OA_2A_1 + \angle OA_2A_3$

$= \theta + \theta$

$= 2\theta$

Therefore, in a convex polygon, the measure of each interior angle is 2θ.

$\therefore$ In a convex polygon with 'n' sides, the sum of the measures of the interior angles is $n \times 2\theta = 2n\theta$.

Again, the measure of the angle formed at the center of a convex polygon is 360°.

Then, the sum of the measures of n interior angles and the angles formed at the center $= 2n\theta + 360°$.

Besides, the sum of the three angles, ΔOA_1A_2 is 180°.

$\therefore$ The sum of the interior angles of n triangles = $n \times 180°$.

Now, the sum of the measures of n interior angles and the angles formed at the center = the sum of the measures of the interior angles of n triangles.

$\therefore 2n\theta + 360° = n \times 180°$

or, $2n\theta = n \times 180° - 360°$

or, $\theta = \dfrac{n \times 180° - 360°}{2n}$

or, $\theta = \dfrac{n \times 180°}{2n} - \dfrac{360°}{2n}$

$\therefore \theta = 90° - \dfrac{180°}{n}$

Now, $\tan\theta = \dfrac{OP}{PA_2} = \dfrac{h}{\frac{a}{2}} = \dfrac{2h}{a}$ [since, $PA_1 = PA_2 = \dfrac{a}{2}$]

or, $\tan\theta = \dfrac{2h}{a}$

or, $2h = a \tan\theta$

$\therefore h = \dfrac{a}{2} \tan\theta$

Now, the area of, $\Delta OA_1A_2 = (\frac{1}{2} \times \text{base} \times \text{height})$ square units

$= \frac{1}{2} \times A_1A_2 \times \text{h}$

$= \frac{1}{2}\, a.\, \frac{a}{2}\ \tan\theta$ [as $h = \frac{a}{2}\ \tan\theta$]

$= \frac{a^2}{4}\ \tan\left(90° - \frac{180°}{n}\right)$ [as $\theta = 90° - \frac{180°}{n}$]

$= \frac{a^2}{4}\ \cot\left(\frac{180°}{n}\right)$ [as $\tan(90° - \alpha) = \cot\alpha$]

Area of a polygon $A_1\, A_2\, A_3\, A_4\, A_5 \ldots A_n = n \times \Delta OA_1A_2$

$= \frac{na^2}{4}\ \cot\left(\frac{180°}{n}\right)$

$\therefore$ The area of a convex polygon with n number of sides, each with a length a

$= \frac{na^2}{4}\ \cot\left(\frac{180°}{n}\right)$ square units

Problem 1: the lengths of the base edges of a triangular prism, are respectively 7 cm, 8 cm, and 10 cm, and the height is 17 cm. Determine the total surface area of the prism.

Problem 2: The lengths of each base edge of a regular pentagonal prism are 3.5 m, and the height of the prism is 12 m. Calculate the lateral surface area of the prism and its total surface area.

Volume of prism

The volume of a solid is an interesting concept. When analyzing various types of solids, it can be observed that those with uniform perimeters, meaning that they have the same circumference on all sides, possess an interesting property. Such solids' volumes can be obtained by multiplying their base area by their height. For instance, rectangular solids, cubes, cylinders, etc. fall into this category. A prism is also a type of solid with a uniform perimeter, where the base has the same circumference on all sides. Therefore, for a prism as well, its volume can be determined by multiplying its base area by its height.

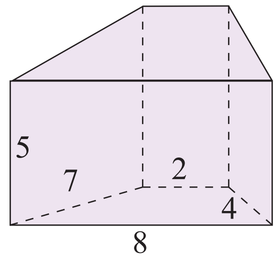
$\therefore$ Volume of the prism = (Base area × Height) cubic units

$\therefore$ Volume of the prism = (Base area × Height) cubic units

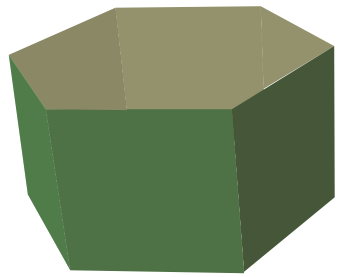
Problem 3: The lengths of the base edges of a triangular prism are 5 cm, 12 cm, and 13 cm, and the height is 41 cm. Determine the total surface area and the volume of the prism.

Problem 4: A regular quadrilateral prism has each base edge measuring 12 cm, and the height of the prism pole is 17 m. Calculate the total surface area and the volume of the prism pole.

Problem 5: In the diagram, the lengths of the diagonals of an irregular vertical quadrilateral prism are given in meters. Determine the total surface area and the volume of the prism.



Problem 6: In the following regular prism-shaped aquarium, each base edge measures 15 cm, and the height is 18 cm. If it costs 20 taka per square centimeter to paint the outside of the aquarium, how much money will be needed in total? Again, if the aquarium is to be filled with colored water at a cost of 15 taka per cubic centimeter, how much money will be needed in total?



Pyramid

All of us are familiar with the secrets of ancient Egyptian pyramids In our personal and social lives, we use so many things every day. The image depicts a few everyday objects. Can you tell what they are used for? You can see a slice of watermelon in the first image. We usually cut watermelons like this before we eat. You can see a tent in the second image. Usually, for a short time stay, we create a temporary house using materials like bamboo, wood, or thick fabric.



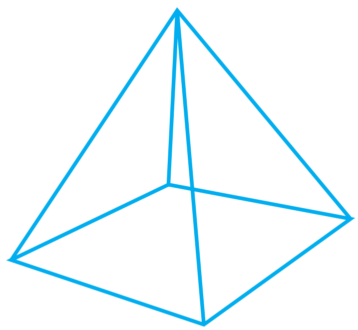

What is seen in the third image is used to denote causation or caution. For example, people are alerted as the place is wet and slippery. And in the fourth figure, the specific location on the road is marked with a prohibition sign, meaning it is forbidden to use. However, they are all solids. In this way, there are many such things that we use in our daily lives.

However, a special characteristic of the objects presented above is that the base of each object is a triangle, quadrilateral, or any other polygon. And each of the lateral surfaces is triangular. Additionally, these lateral surfaces converge at a single vertex. Can you tell the name of this type of solid? The name of this type of solid is a **pyramid**.

Can you create a list of a few more names of objects or solids of this type? Combine your concepts and write a list of a few more objects' names similar to the images above.

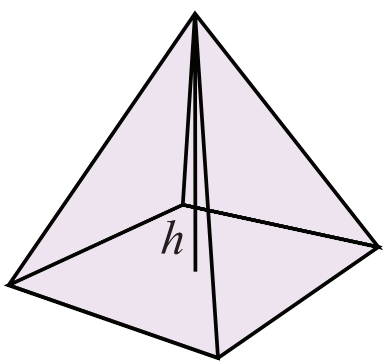
Serial no.	1	2	3	4
Name of object				

Pyramid: A pyramid is a three-dimensional solid located above a polygonal base. The lateral faces of a pyramid converge at a single vertex. However, the base of a pyramid can be of any polygonal shape, and its lateral faces can be of any triangular shape.



A pyramid is called a **"regular pyramid"** or **"symmetrical pyramid"** if its base is an equilateral polygon and all its lateral faces are congruent triangles. Regular pyramids are particularly visually pleasing.

If the base of a pyramid is not an equilateral polygon and its lateral faces are not congruent triangles, it is referred to as an **"irregular pyramid"** or **"asymmetrical pyramid."**



If a perpendicular drawn from the apex of a pyramid to its base meets the base at the center, it is called **a right pyramid**. The center of the base is referred to as the incenter of the base polygon. The point where the hypotenuses of a regular pyramid intersect is called the incenter. A regular pyramid can be considered a type of right pyramid.

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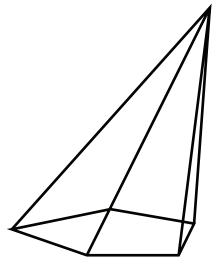
If a perpendicular drawn from the apex of a pyramid to its base does not meet the base at its center, it is called **an oblique pyramid**.



Apex of the Pyramid: The point where the lateral faces of the pyramid converge at a common point is called the apex of the pyramid.

Base of the Pyramid: The plane figure on which the pyramid is situated is referred to as the base of the pyramid. The base of the pyramid can take the shape of any polygon.

Height of the Pyramid: The length of the perpendicular drawn from the apex of the pyramid to the base is known as the height of the pyramid.

Slant Height of the Pyramid: The length of the perpendicular drawn from the apex of the pyramid to a lateral face's base is called the slant height of the pyramid.

Pyramid's Apothem: The line segment connecting the apex of the pyramid and any angular point on the base of the pyramid is called the pyramid's apothem.

In any pyramid, the number of sides in the base polygon is equal to the number of lateral faces. However, the number of edges in the apothem is twice the number of sides in the base polygon or the number of lateral faces.

So, by analyzing a pyramid, the following elements can be found:

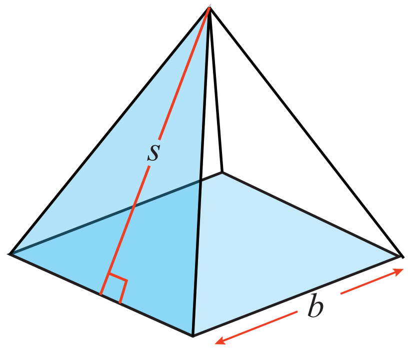
- One apex/vertex
- One polygonal base
- At least three or more triangular lateral faces

Area of pyramid

Now, let us try to calculate the surface area of a pyramid. The sum of the base and the surface areas of the lateral faces constitutes the total surface area of the pyramid.

Area of base

The area of the base on which a pyramid stands is the area of the pyramid. If the base of the pyramid is triangular, quadrangular, pentagonal or any other polygon, its area must be determined first.



Lateral surface area

The lateral faces of a pyramid are triangular. Let us assume the length of the base sides of this quadrangular pyramid is a, b, c, d units respectively and the slant height is s units. It's important to note that the height of each triangle is the slant height.

$\therefore$ Lateral surface area of the pyramid $= \frac{1}{2} a \times s + \frac{1}{2} b \times s + \frac{1}{2} c \times s + \frac{1}{2} d \times s$

$= \frac{1}{2}(a + b + c + d) \times s$

$= \frac{1}{2}$ (Base Perimeter × Slant Height) square units

This formula for the lateral surface area of a pyramid is applicable to all pyramids

$\therefore$ Lateral surface area of a pyramid $= \frac{1}{2}$ (Base Perimeter × Slant Height) square units

$\therefore$ Total surface area of the pyramid = Base area + Areas of lateral faces

$\therefore$ Total surface area of the pyramid = Base area + $\frac{1}{2}$ (Base perimeter × Slant height) square units

$\therefore$ Total surface area of the pyramid = Base area + $\frac{1}{2}$ (Base perimeter × Slant height) square units

Problem 1: The lengths of the base's sides of a triangular pyramid are 3 cm, 4 cm, and 5 cm respectively, and the height is 13 cm. Determine the total surface area of the pyramid.

Problem 2: In a right hexagonal pyramid similar to a minaret, each side of the base measures 4.8 m, and the height of the minaret is 15 m. Calculate the lateral surface area and the total surface area of the minaret.

Volume of pyramid

The volume of a pyramid can be obtained by multiplying the base area of the pyramid by its height and then dividing the result by 3

$\therefore$ Volume of the pyramid $= \frac{1}{3}$ (Base area × Height) cubic units.

Problem 3: Consider a triangular pyramid with base sides of lengths 28 cm, 45 cm, and 53 cm, and a height of 62 cm. Determine the total surface area and volume of this pyramid.

Problem 4: In a smooth octagonal pyramid, each base side has a length of 23 cm, and the height of the pyramid is 3.7 meters. Calculate the total surface area and volume of the pyramid.

Exercise

1. 1. A 12 cm long rectangular carrot tip has a diameter of 2.5 cm. To eat the carrot, you peel the carrot. What is the area of all the shells? Determine the nutritional content of the carrot.

2. As shown in the picture, the total surface area of a cuboid made of plastic found on the road is 1256.64 square cm, and the slant height is 26 cm.

   (i) If it costs 1.50 taka per square centimeter to paint the curved surface of the cuboid with red color, how much will it cost in total?

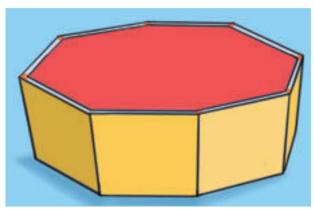
   (ii) How much plastic is there in the cuboid?

3. A plastic solid sphere has a radius of 6 cm. If the sphere is melted into a hollow sphere of radius 7 cm, find the plastic density of the hollow sphere.

4. The radii of four perfect spheres are 3 cm, 8 cm, 13 cm and r cm respectively. What is the value of r if four spheres are melted to form a new solid sphere of radius 14 cm?

5. An equilateral heptagonal prism shaped aquarium has each side of length 25 cm and height 1 m. If it costs 2 taka per square centimeter to cover the lateral surface of the aquarium with glass, how much will it cost in total? How many liters of water will be required to fill three-fourths of the aquarium, given that 1000 cubic centimeters = 1 liter?

6. Each side of the plane of the equilateral prism in the figure is 5 cm and the sides are square.



   (i) Measure the surface area of the prism.

   (ii) What is the curved surface area of the prism?

   (iii) Determine the volume of the prism.

7. A tent is constructed on a square ground of length 8 $\sqrt{2}$ m by placing a pole of $\sqrt{66}$ high right in the middle.

   (i) Determine the length of the diagonal of the prism.

   (ii) How much money needs to be spent to purchase fabric at the rate of 200 taka per square meter?

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(iii) Calculate the volume of the empty space inside the prism.

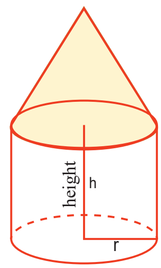
8. A pyramid with a base of a square of side 6 meters and a slant height of $\sqrt{67}$ meters is situated on the square land.

   (i) Determine the height of the pyramid.

   (ii) What is the total surface area of the pyramid?

   (iii) Calculate the volume of the pyramid.

9. The given solid in the figure has a base with a radius of 4 meters in the lower part and a height of 5 meters. The sloping height in the upper part is 3 meters.

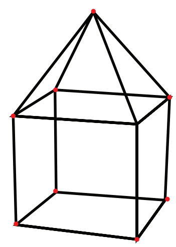


   (i) If it costs 450 taka per square meter to paint the curved surface of the solid in the lower part, how much will it cost in total?

   (ii) What is the total surface area of the solid?

   (iii) Determine the volume of the solid.

10. The solid given in the figure is placed on a rectangular base with dimensions of length 6 meters and width 4 meters. The height of the lower part is 7 meters. The length of the diagonal in the upper part is 7.5 meters.

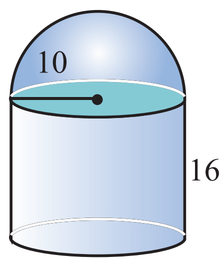


   (i) If it costs 2250 taka per square meter to apply iron sheets on the lateral surface of the lower part of the solid, how much will it cost in total?

   (ii) Determine the lateral surface area of the upper part of the solid.

   (iii) Determine the volume of the solid.

11. The solid given in the figure has a base radius of 10 centimeters and a height of 16 centimeters in the lower part.



   (i) What is the height of the geometric solid?

   (ii) Determine the surface area of the upper part of the geometric solid.

   (iii) Calculate the total surface area of the geometric solid.

   (iv) Determine the volume of the geometric solid.

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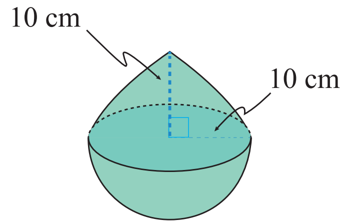
12. Notice the geometric solid shown in the figure carefully.



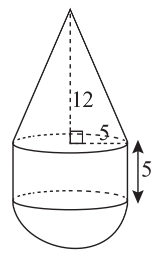
(i) What is the length of the inclined base of the solid?

(ii) Determine the surface area of the upper part of the geometric solid.

(iii) Determine the total surface area of the geometric solid.

(iv) Determine the volume of the geometric solid.

13. Notice the geometric solid shown in the figure carefully.



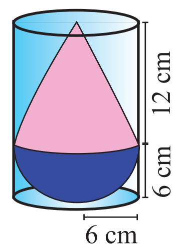
(i) Determine the surface area of the upper part of the geometric solid.

(ii) Determine the height of the solid.

(iv) Determine the total surface area of the geometric solid.

(v) Determine the volume of the geometric solid.

14. In the figure, a hemisphere and a cone are placed accurately inside a cylinder.



(i) Calculate the lateral surface area of the cone.

(ii) Find the area of the curved surface of the hemisphere.

(iii) Determine the volume of the hollow part inside the cylinder.

(iv) What is the ratio of the volumes of the hemisphere, the cone, and the cylinder?

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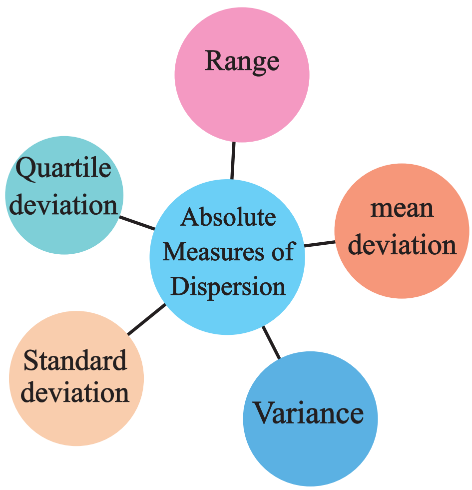
Measures of Dispersion

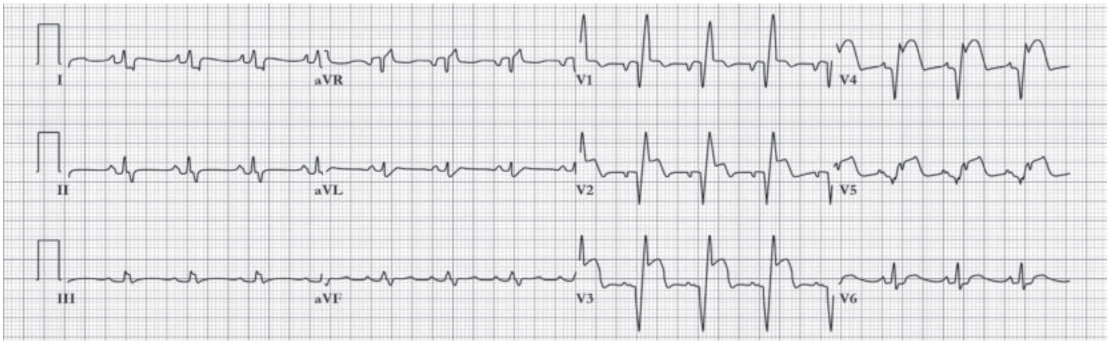
You can learn from this experience-

- Prior experience and its reflection
- Necessity of measures of dispersion in objective decision making
- Types of measures of dispersion
- Measuring different types of dispersion
- Which kind of measurement is more reliable in individual decision making



Measures of Dispersion

You have already come to know that statistics works with the data collected for its own definite purposes. We take decisions based on the analyses and explanations of the collected data. In the previous class, you have known about the graphical representation of data. Such kind of representation reveals some important features of the data. Note the image below.



For checking our heart rate and rhythm, doctors do an electrocardiogram, or ECG, whose graph looks almost like this. By looking at such graphs, doctors diagnose heart attacks, heart diseases, abnormal heartbeats etc. and give prescriptions. Also, you have consistently learned in previous classes about measures of central tendency to find values that represent the data collected. When it comes to decision-making, central values give us a rough idea. But more accurate conclusions require careful analysis and interpretation of the data. In that case, we also need to know how the data are scattered around the central value.

Let us discuss the matter through an example –

As you know, every year district wise "T - 20 School Cricket" competition is organized with schools of your district. As usual this time as well your school cricket team participates in that competition.

Suppose, the runs scored by two batsmen A and B of your school in ten matches in a competition and the final match's batsmen's (frequency) table are as follows:

Batsmen A: 30, 91, 0, 64, 42, 80, 30, 5, 117, 71

Batsmen B: 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

Suggestions

Make two teams with the learners of your class. Organise some cricket matches with them. Then collect the scores of any two or three batsmen and bowlers.

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Frequency table of the final match

Over	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
Run	24	32	36	38	30

Let us find batsman A's average run:

$\overline{x}=$

$=$

$= \boxed{53}$

Let us find batsman B's average run:

$\overline{x}=$

$=$

$= \boxed{}$

Let us find the median of the run of batsman A

Let us arrange the data in ascending order,

$=$

$=$

$= \boxed{}$

Let us find the median of the run of batsman B

Let us arrange the data in ascending order,

$=$

$=$

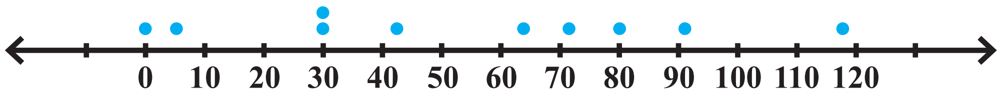
$= \boxed{53}$

What value do you get by determining the mean and median of the runs scored by both batsmen? Average and median of the runs of both the batters are same. Do you think that these two players have the same skills? Not at all. Because in an equal number of matches, batsman A's run range is (0-117) and batsman B's run range is (46-60). Can you notice any difference in the consistency of skill of two batsmen? If yes, briefly write your opinion with reasons in the blank box below:

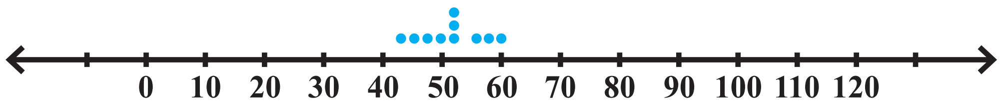
Brain storm 

Let's plot the scores of both batsmen through the dots on the number line to test your hypothesis.

A numerical representation of batsman A's scores on the number line.



A numerical representation of batsman B's scores on the number line.



By observing the above two figures, we can see that the score points of batsman B are very close to the central values (mean and median). On the other hand, batsman A's score points are scattered far away from the central value (mean and median), even though their runs average and median are the same.

So we can say, the measurement of the central tendency of the collected data is not enough to take an objective decision on any issue. The dispersion of the data relative to the central value should also be measured because it verifies the correctness of the central standard. The smaller the dispersion of the data series, the more representative the central values are. Dispersion measures the consistency of data values. The wider the dispersion of data are, the more inconsistent the values are.

Therefore, in this chapter we will know about the importance and methods of determination of **Measures of Dispersion**.

As you must have understood from the above discussion, the dispersion is the difference between the central value and the other values of the data. It gives an idea of how far other values are from the central value of the data. According to A.L.Bowley, "Dispersion is the measures of the variation of the items", that is, dispersion is a measure of the variance of the elements of a data series.

So, the mathematical measure which measures the distance from the central value of a measurement to other values can be called spread measure.


Statistician A. L. Bowley

As you already know, the main purpose of measuring dispersion is to compare two or more data series. Different types of measurement of dispersion are used based on nature of the data series. However, in this class we will try to briefly learn about range,

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mean deviation, variance and standard deviation from different types of measurement of dispersion.

Range

Range is the difference between the largest value and the smallest value of a data set. However, in the case of categorical data, the range is the interval between the upper limit of the last class and the lower limit of the first class. The range is usually denoted by R.

Individual task:

Measure the height of your family members in centimeters or inches to determine the range.

Brain storm

Range is always positive, why?

____

____

In case of unstructured or uncategorized data:

The n number of values of a variable x are $x_1, x_2, x_3, \ldots, x_n$ respectively. Take the smallest value of them which is x_L and the largest value of them which is x_H. So the range is - $R = x_H - x_L$ or, $R = |x_H - x_L|$

In case of structured or classified data:

Let us assume L_u be the upper limit of the latest class and L_l the lower limit of the lowest class

So range is - $R = L_u - L_l$

Instructions

If the mass number of the first class of a population is 0 (zero), the next class after the first class should be treated as the first class in determining the range. Then determine the range.

Individual task: 1

a) Determine the range of the following set of data −12, −7, −2, 0, 7, 8.

b) Suppose, the class wise attendance list of 62 students in your class in an average month was as follows.

Number of attendance	1 – 3	4 – 6	7 – 9	10 – 12	13 – 15	16 – 18
Number of students	2	3	7	12	30	8

Do the calculations in your own notebook and write the results in the box below:

- a) the range (R) that get after calculation, is =
- b) the range (R) that get after calculation, is =

Single task: 2

The tallest flowering plant in your garden is 75.06 cm tall and the plants range in height from 15.37 cm. Find the height of the smallest flower.

Range of first 15 prime numbers

The role of Range in decision making

A very simple method is to measure dispersion by determining the range. Its value can be determined in a very short period of time. Only the maximum and minimum values of the data are used to determine the range and all other values are ignored. Not only that, but the range cannot be measured if the lower and upper limits of marginal classes are free in categorical data. But it instantly gives an overall idea of the spread of a dataset. In practice, the mean interval is much more commonly used as a measure of spread than the range.

The use of Range in our daily life

As we know, range is not a representative measure of dispersion. So it is not widely used in practical life. However, the use of range in certain cases is undeniable. For example:

(i) We listen to the weather forecast on radio or television every day. If you notice, you will see or hear that meteorologists talk about the maximum and minimum temperature instead of the average temperature when describing the daily temperature. That is, they use a range of data.

(ii) Many of you must have heard about the stock market. In the stock market, the prices of shares go up and down continuously. So both share buyers and sellers need to know the minimum and maximum price range of the share. If the share price range is known, there is less chance of losses for share buyers and sellers in the bargain.

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Group Work

(i) Divide yourselves into groups and measure the height (in inches) of all the students in your class. Determine the range of data obtained.

(ii) Classify the data with appropriate class intervals. Now determine the range of hierarchical data.

Mean Deviation

Mean interval is a type of dispersion measure that measures the distance from the mean of each value in a data series. That is, to measure how far the data is on an average from the central value of the data series. We know, "The sum of the deviations of each value of the data from their arithmetic mean is zero." So if the positive and negative signs are taken into account while measuring the interval, then measuring the average interval is completely meaningless. Therefore, the absolute value of the interval is considered without taking into account the signs when measuring the interval from each value. The mean difference is the value that we get from the sum of the absolute values of the differences of numbers between the mean, the median and the mode in a chart/ record which is divided by the total population.

Let's calculate if the sum of the deviations from the arithmetic mean of the data values is zero or not.

Suppose, there are 5 members in your family whose ages (in years) are 5, 12, 36, 40 and 67 respectively. Average age of family members $\overline{x} = 32$

$\therefore$ The sum of the deviations from the mean $\sum_{i=1}^{5}(x_i - \overline{x})$

$= (5 - 32) + (12 - 32) + (36 - 32) + (40 - 32) + (67 - 32)$

$= -27 - 20 + 4 + 8 + 35 = 47 - 47 = 0$

So we can say, "The sum of the deviations of each value of the data from their arithmetic mean is zero."

Brain storm

- Show that the mean interval determined from the median is the smallest.
- Prove that the mean interval of two unequal data is half of their range.

Individual task:

Find the sum of the runs scored by batsmen A and B and their arithmetic mean deviation.

Determining the mean deviation for unstructured or uncategorized data

Suppose, the number of absent students in your class for the last eight days is: 3, 6, 6, 7, 8, 11, 15, 16

We have to do three things to find the mean deviation of the data series

What to do in determining the mean deviation:

- First find the mean of the data series
- Finding the variance of each data set from the deterministic mean
- Averaging the differences

Stage - 1: First find the mean of the numbers on absent students

Average of numbers

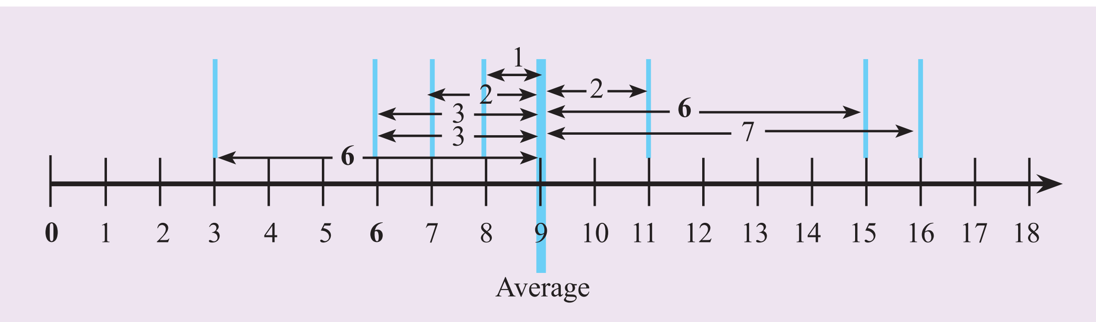
$$= \frac{3+6+6+7+8+11+15+16}{8}$$

$$= \frac{72}{8} = 9$$

Stage - 1: Find the difference of each value of the data from the mean number 9:

Numbers on absent students	3	6	6	7	8	11	15	16
Difference of each value from mean 9	6	3	3	2	1	2	6	7

Let's visualize and try to understand the difference of each value from the mean



Step – 3: Now let us determine the average from the difference of each value from mean 9.

mean deviation

$$= \frac{6+3+3+2+1+2+6+7}{8}$$

$$= \frac{30}{8} = 3.75$$

Brain storm

In the case of a mean interval, the sum of the intervals to the left and right of the mean will be equal. The figure averages 9. Check the correctness of the statement.

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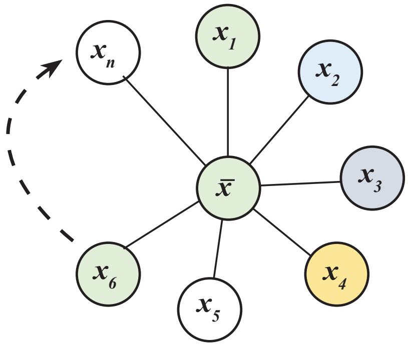
So, the mean of the number of absent students in your class for the last eight days is 9 and the mean deviation is 3.75. By calculating the mean deviation, you can understand how far other values are from the mean.

Calculate the mean deviation for unstructured or uncategorized data by the formula:

Consider that any x variable has n number of values $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ respectively.

So to find the mean interval of the data series we can follow the following steps:



Step – 1: Finding the arithmetic mean $\bar{x}$ of the data.

Step – 2: Finding the interval of $\bar{x}$ from each value in the data.

Step – 3: Determine the absolute value of the interval of $\bar{x}$ from each value in the data.

For example: $|x_1 - \bar{x}|, |x_2 - \bar{x}|, |x_3 - \bar{x}|, \ldots\ldots |x_n - \bar{x}|$.

Step – 4: Finding the mean of n number of intervals.

It means, the mean $= \dfrac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$

Following these four steps, the n number of deviations that we have determined is called the formula of "Mean deviation determined from arithmetic mean" of unstructured or uncategorized data.

So,
Mean deviation determined from arithmetic mean,

$$\text{M. D}(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

Similarly, the deviation of unstructured data, determined from the median (M_e), is called the "mean deviation determined from the median" and when determined from the range (M_o), it is called the "mean deviation determined from the mode".

$\therefore$ Mean deviation from the median, M.D (M_e) $= \dfrac{\sum_{i=1}^{n} |x_i - \text{M}_e|}{n}$

$\therefore$ Mean deviationfrom mode, M.D (Mo) $= \dfrac{\sum_{i=1}^{n} |x_i - \text{M}_o|}{n}$

Example - 1: Let's calculate the mean deviation from the arithmetic mean of runs scored by batsman A using the formula:

Step – 1: Arithmetic mean of runs scored by batsman A, $\overline{x} = 53$ [You have already determined]

Step – 2: Find the difference ($|x_i - \overline{x}|$) of each value of the data from the arithmetic mean, $\overline{x} = 53$

Run (x_i)	$\lvert x_i - \overline{x}\rvert$
30	23
91	38
0	53
64	11
42	11
80	27
30	23
5	48
117	64
71	18
$\sum\lvert x_i - \overline{x}\rvert = 316$	

Stage – 3:

$\therefore$ Mean deviation M.D($\overline{x}$)

$$= \frac{\sum_{i=1}^{n}|x_i - \overline{x}|}{n} = \frac{316}{10}$$

$= 31.6$

That is, the difference in runs scored by batsman A from the arithmetic mean is 31.6. In this case we can say that the gap is very high, the consistency of batsman A's skill is low.

Individual task:

a) Find the mean deviation between the median and the mode of the runs scored by batsman A.

b) Find the arithmetic mean, median and mode of the runs scored by batsman B.

c) Review the mean difference of data obtained from batsmen A and B and present your opinion about their abilities.

Determining the mean deviation of categorical data :

Consider, the midpoints of n number of classes of a mass distribution are x_1, x_2, x_3,........., x_n and their periodic mass numbers are $f_1, f_2, f_3, \ldots\ldots\ldots, f_n$ respectively.

First, the arithmetic mean ($\overline{x}$), median (Me) and mode (Mo) of the categorical data should be determined. Then the mean deviation of the data can be calculated using the formula below.

Formula for determining the mean deviation of categorical data:

(i) Mean deviation determined from arithmetic mean

$$= \frac{\sum f_i\,|x_i - \overline{x}|}{n}$$

(ii) Mean deviation determined from the median

$$= \frac{\sum f_i\,|x_i - \mathrm{M_e}|}{n}$$

(iii) Mean deviation determined from the mode

$$= \frac{\sum f_i\,|x_i - \mathrm{M_o}|}{n}$$

In previous classes I learned to calculate the arithmetic mean ($\overline{x}$), median ($\mathrm{M_e}$) and extreme ($\mathrm{M_o}$) of categorical data

Example 2: Let's calculate the arithmetic mean to determine the mean deviation using the chart/ table of the final match:

Over	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
Run	24	32	36	38	30

Arithmetical average,

$$\bar{x} = a + \frac{\sum f_i u_i}{n} \times h$$
$$= 10.5 + \frac{18}{160} \times 4$$
$$= 10.5 + 0.45$$
$$= 10.95$$

Suppose, assumed average, $a = 10.5$

Class interval $h = 4$

Total run , $n = 160$

$\sum f_i u_i = 18$

For calculating the mean deviation, let us first make the following table:

class interval (Over)	Midpoints of classes (x_i)	Run (f_i)	Step deviation $u_i = \frac{x_i - a}{h}$	$f_i u_i$	$x_i - \bar{x}$ $\bar{x} = 10.95$	$f_i \lvert x_i - \bar{x} \rvert$
1 – 4	2.5	24	–2	–48	8.45	202.8
5 – 8	6.5	32	–1	–32	4.45	142.4
9 – 12	10.5 = a	36	0	0	0.45	16.2
13 – 16	14.5	38	1	38	3.55	134.9
17 – 20	18.5	30	2	60	7.55	226.5
		$n = 160$	$\sum f_i u_i = 18$		$\sum f_i \lvert x_i - \bar{x} \rvert = 722.8$	

So mean deviation determined from arithmetic mean $= \frac{\sum f_i \lvert x_i - \bar{x} \rvert}{n} = \frac{722.8}{160} = 4.52$ (approx.)

Example 3: Calculate the difference between the median and the mean using the table of the final match:

Class Range (Over)	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
Run	24	32	36	38	30

Median, $M_e = L + (\frac{n}{2} - F_c) \times \frac{h}{f_m}$

$= 9 + (80 - 56) \times \frac{4}{36}$

$= 9 + 2.67$

$= 11.67$ (approx.)

Here,

$L = 9$, $F_c = 56$,

$f_m = 36$, $h = 4$

and $n = 160$

class interval (Over)	Midpoints of classes (x_i)	Run (f_i)	Number of sequential runs (F_c)	$\lvert x_i - M_e \rvert$ $M_e = 11.67$	$f_i \lvert x_i - M_e \rvert$
1 – 4	2.5	24	24	9.17	220.08
5 – 8	6.5	32	56	5.17	165.44
9 – 12	10.5	36	92	1.17	42.12
13 – 16	14.5	38	130	2.83	107.54
17 – 20	18.5	30	160	6.83	204.9
		$n = 160$	$\sum f_i \lvert x_i - M_e \rvert = 740.08$		

Here, $\frac{n}{2} = \frac{160}{2} = 80$ ∴ median will be the 80^{th} term. As 80^{th} term is in the range (9 – 12), the median range of the data will be (9 – 12).

Hence the mean deviation from the median $= \frac{\sum f_i \lvert x_i - M_e \rvert}{n} = \frac{740.08}{160} = 4.62$ (approx.).

Group Work: Divide all the students of your class into groups and measure each group's height (cm). Now construct a hierarchical table of the obtained heights with appropriate class deviation. Using the table, determine mean deviation from (i) Arithmetic mean (ii) Median (iii) Mode respectively.

Standard Deviation


Karl Pearson

Carefully Observe the three data sets given below.

$X = \{12, 7, 6, 5, 4, 3, 2\}$

$Y = \{12, 10, 10, 9, 9, 9, 2, 2\}$

$Z = \{12, 4, 4, 3, 2, 2, 2\}$

It is clear that the range of the above three sets of data is the same and is 10. Besides, the range does not depend on all values of the array. It is affected by extreme values and sample deviations. But a closer look at the above three data sets shows that there are differences in numbers and even in their central values. Although the mean deviation is dependent on each value in the data series, it is again determined by the absolute value of the deviation. Hence it cannot be used in any subsequent algebraic process. Moreover, since negative deviations are mathematically treated as positive to obtain the absolute value, it is prone to error in many cases. Therefore, a new measure is needed to resolve the true variance among all values in the data series. In this case, standard deviation plays a more effective role. In 1983, Karl Pearson introduced the concept of standard deviation.

But before knowing the standard deviation we need to know about Variance.

Variance

You must have remembered that the absolute value of the deviation was used to determine standard deviation from arithmetic mean or median. But why? Briefly write the reasons in the box below.

Brain storm

We can also solve the problem of using absolute values in determining the mean deviation in another way. Square the deviation between each value of the data and their mean or median. In that case the square of deviation must be non-negative.

Academic year 2024

Suppose, a variable x have n number of values $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$, respectively and their arithmetic mean $\overline{x}$. Then, the equation will be, $(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2$. If the sum of the squares of the deviation is zero, then $(x_i - \overline{x})$ must be zero. In that case there will be no difference between mean and values. But if the value of $\sum_{i=1}^{n}(x_i - \overline{x})^2$ is very small, each value in the data series will be very close to the arithmetic mean or central value. That is, the deviation of dispersion will be less. In that case we can say, the values of the dataset are much more consistent.

Let's try to understand the matter through an example:

Aleya's family has 6 members and their ages are 5, 15, 25, 35, 45, 55 years respectively. Average age of family members $\overline{x} = 30$. [Check the calculation]

In this case, the arithmetic mean is the sum of the squared deviations of each value from $\overline{x}$

$$\sum_{i=1}^{6}(x_i - \overline{x})^2 = (5 - 30)^2 + (15 - 30)^2 + (25 - 30)^2 + (35 - 30)^2 + (45 - 30)^2 + (55 - 30)^2$$
$$= (-25)^2 + (-15)^2 + (-5)^2 + (5)^2 + (15)^2 + (25)^2$$
$$= 625 + 225 + 25 + 25 + 225 + 625 = 1750$$

On the other hand, Thomas's family is a close family. There are a total of 31 members in the family. There is always a festive atmosphere at home. The ages of Thomas' family members are 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37 respectively. , 38, 39, 40, 41, 42, 43, 44, 45 years and the mean of their ages is $\overline{y} = 30$. [Check the calculation in this case also]

$$\therefore \sum_{i=1}^{31}(y_i - \overline{y})^2 = (15 - 30)^2 + (16 - 30)^2 + (17 - 30)^2 + \ldots + (45 - 30)^2$$
$$= (-15)^2 + (-14)^2 + (-13)^2 + \ldots + 15^2$$
$$= 2\,(1^2 + 2^2 + \ldots + 15^2)$$
$$= 2 \times \frac{15 \times (15 + 1)(30 + 1)}{6}$$
$$= 5 \times 16 \times 31 = 2480$$

For all natural numbers n

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

By reviewing the two calculations given above, it can be seen that the arithmetic mean of the ages of the family members of Aleya and Thomas is the same. But the age dispersion of Aleya's family members (50) is greater than the age dispersion of Thomas's family members (30).

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So we can say, in the case of dispersion measurement, the problem will not be solved only by finding the sum of the squared data of the data series and their arithmetic mean. We need to determine the mean of the sum of the squares of the data in the data series and their arithmetic mean.

It means we need to determine $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$

For example, in the case of Aleya's family, we will get: $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{6} \times 1750 = 291.67$

And in the case of Thomas's family, we will get: , $\frac{1}{n}\sum_{i=1}^{n}(x_\text{i} - \overline{x})^2 = \frac{1}{31} \times 2480 = 80$

It is also clear from the results of the two families that the average age of Thomas's family members has a much larger age dispersion than that of Aleya's family members.

Therefore, $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$ can be a suitable medium to measure dispersion. And this is **Variance**. It is denoted by the symbol σ^2 (read sigma square).

So the value obtained by dividing the sum of the squares of the deviation of the arithmetic mean from each value of a data series by the total number of data is called Variance.

Brain storm

Age variance of Aleya and Thomas family members respectively

____________________________________ and

_____________________________________.

Let us determine Variance through formula :

In the case of unstructured or uncategorized data:

If a variable x has n number of values $x_1, x_2, x_3, \ldots, x_n$ and their arithmetic mean is $\overline{x}$ respectively,

variance is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$

In the case of structured or categorized data:

A variable x has n number of values $x_1, x_2, x_3, \ldots, x_n$ respectively whose mass numbers are $f_1, f_2, f_3, \ldots, f_n$ respectively. Now if the arithmetic mean is $\overline{x}$,

Variance is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} f_i(x_i - \overline{x})^2$

Example 4: Let's calculate the Variance of runs scored by batsman B using the formula:

Step – 1: Arithmetic mean of runs scored by batsman B, $\overline{x} = 53$ [You have already determined]

Step – 2: Calculate the difference $x_i - \overline{x})^2$ of each value of the data from the arithmetic mean, $\overline{x} = 5$:

Run x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
53	0	0
46	-7	49
48	-5	25
50	-3	9
53	0	0
53	0	0
58	5	25
60	7	49
57	4	16
52	-1	1
$\sum_{i=1}^{10}(x_i - \overline{x})^2 = 174$		

Step – 3

$$\therefore \text{Variance } \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} f_i(x_i - \overline{x})^2$$

$$= \frac{174}{10}$$

$$= 17.4$$

$\therefore$ Variance 17.4

Instructions

Unstructured or uncategorised data does not need to be sorted to determine the variance.

Let us create a formula from another formula

Already you have learnt how to determine variance using this formula $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$.

What if we could simplify this formula a bit and make it more user-friendly? So let's try:

$$\text{Variance } \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2x_i\overline{x} + \overline{x}^2)$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\overline{x}.\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\overline{x}^2$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\overline{x}.\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}(\overline{x}^2 + \overline{x}^2 + \overline{x}^2 + \dots\ n \text{ times } \overline{x}^2)$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\overline{x}.\overline{x} + \frac{1}{n}.n.\overline{x}^2$$

Brain storm

In which case Variance will be the lowest?

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\overline{x}.\overline{x} + \overline{x}^2$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\overline{x}^2 + \overline{x}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \overline{x}^2$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2$$

Brain storm

Find the arithmetic mean and standard deviation of the first n numbered natural numbers.

$\therefore$ **Variance** $\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2$. What is the reason of considering the use of this formula relatively simple in determining Variance? Express your opinion in one or two lines.

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Pair work:

Toss a five taka coin 20 times. Write down the number of times you received Shapla in the notebook. Play the game together 10 times in total. Suppose the number of Shapla in coin toss is 6, 8, 10, 10, 10, 11, 12, 12, 13, 13. Determine the arithmetic mean and standard deviation of the data obtained.

Example 5: Calculate the mean and variance using the final match table:

Class Range (Over)	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
Run	24	32	36	38	30

First create the following table for determining variance:

Deviation of classes (Over)	Midpoints of classes (x_i)	Run (f_i)	$f_i x_i$	$(x_i - \overline{x})^2$ $\overline{x} = 10.95$	$f_i (x_i - \overline{x})^2$
1 – 4	2.5	24	60	71.4025	1713.66
5 – 8	6.5	32	208	19.8025	633.68
9 – 12	10.5	36	378	0.2025	7.29
13 – 16	14.5	38	551	12.6025	478.895
17 – 20	18.5	30	555	57.0025	1710.075
		$n = 160$	$\sum f_i x_i = 1752$	$\sum f_i (x_i - \overline{x})^2 = 4543.6$	

$\therefore$ Variance $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} f_i (x_i - \overline{x})^2$

$= \frac{4543.6}{160}$

$= 28.40$ (approx.)

Here,

Arithmetic mean, $\overline{x} = \frac{\sum f_i x_i}{n}$

$= \frac{1752}{160} = 10.95$

Group work:

(i) Divide yourselves into groups and measure the weight (in kg) of all the students in your class. Determine the variance of the obtained data.

(ii)Classify the data with appropriate class deviation. Now find the variance in hierarchical data in a simple way.

Standard Deviation

What is Standard deviation?

The square root of the value obtained by dividing the sum of the squares of the arithmetic mean of the values of a data series by the total number of terms is called the standard deviation. That is, the square root of the variance (σ^2) is the s**tandard deviation**. The standard deviation is expressed by σ (Greek letter sigma) or SD.

Where and why do we use standard deviation?

You will be surprised to know that there are many examples in our daily life where we are applying mathematical phenomena like standard deviation without knowing it. For example:

1. We allocate an average amount in our daily budget according to our income and needs. Without any mathematical calculations, we use a standard deviation to determine whether we are spending too much or too little. This is obviously an instinctive calculation that my mind does for me.
2. Moreover, it is widely used in the analysis of data related to various types of quantitative research, planning, social activities and industrial performance of homogeneous products.
3. Standard deviation is an important tool that business owners use in risk management and decision-making. They use it to develop potential risk management strategies for situations such as declining sales or increasing bad customer reviews.
4. Standard deviation is used in medical research and drug development. You might be thinking, how is it possible? As you know, the discovery of a new vaccine for a new virus like the coronavirus became imperative. For this, the virus is tested with a large number of anti-virals and monitored over time. The average rate of virus

eradication in each sample is calculated by means of standard deviation to see if the antivirals have the same effect.

Determine standard deviation through formula:

In the case of unstructured or uncategorized data

a) Direct method: If n number of values of a variable x are $x_1, x_2, x_3, \ldots, x_n$ respectively and their arithmetic mean $\overline{x}$, the standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}} \quad \text{or,} \quad \sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}$$

b) **Simple or approximate mean or simplified method:** standard deviation,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} d^2}{n} - \left(\frac{\sum_{i=1}^{n} d}{n}\right)^2}$$

Here, A= approximate mean and $d = x - A$

Brain storm

i)What are the mean deviation and standard deviation of these numbers $-2x, -x, 0, x, 2x$?

ii) If the mean and variance of the two expressions are 10 and 4 respectively, find the two expressions.

iii) If the standard deviation of the first n natural numbers is $\sqrt{10}$, what is $n = ?$

In the case of structured or categorized data :

a) Direct method: A variable x has n number of values $x_1, x_2, x_3, \ldots, x_n$ whose mass numbers are $f_1, f_2, f_3, \ldots, f_n$ respectively. Now if the arithmetic mean is $\overline{x}$,

Standard deviation, $\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \overline{x})^2}{n}}$ or, $\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} f_i x_i}{n}\right)^2}$

b) Simple or approximate mean or simplified method: standard deviation,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} fd^2}{n} - \left(\frac{\sum_{i=1}^{n} fd}{n}\right)^2} \times h$$

Here, A = approximate mean, $d = \frac{x - A}{h}$ and h = class deviation

Example 6 : Suppose the weight of 8 students (in kg) of your class are as follows.

49,63,46,59,65,52,60,54

Let us determine the standard deviation of the above data in a simplified way.

Brain storm

i) Why is the standard deviation non-negative?

(ii) In which case, the standard deviation be minimum?

Number (x)	$d = x - A$	d^2
46	–13	169
49	–10	100
52	–7	49
54	–5	25
59 = A	0	0
60	1	1
63	4	16
65	6	36
$n = 8$	$\sum d = -24$	$\sum d^2 = 396$

Suppose, approximate mean A = 59

Here, total mass number $n = 8$

$\sum d = -24$ and

$\sum d^2 = 396$

$\therefore$ standard deviation,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} d^2}{n} - \left(\frac{\sum_{i=1}^{n} d}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{396}{8} - \left(\frac{-24}{8}\right)^2}$$

$$= \sqrt{40.5} = 6.36 \text{ (approx.)}.$$

Example 7 : Let us determine standard deviation in a simplified way using the table of the final match:

Class range (over)	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
Run	24	32	36	38	30

First let us create the following table for standard deviation.

Class deviation (over)	Class middlepoint (x)	Run (f)	$d = \frac{x - A}{h}$	fd	fd^2
1 – 4	2.5	24	–2	–48	96
5 – 8	6.5	32	–1	–32	32
9 – 12	10.5=A	36	0	0	0
13 – 16	14.5	38	1	38	38
17 – 20	18.5	30	2	60	120
		$n = 160$		$\sum fd = 18$	$\sum fd^2 = 286$

$$\therefore \text{Standard deviation}, \sigma = \sqrt{\frac{\sum_{i=1}^{n} fd^2}{n} - \left(\frac{\sum_{i=1}^{n} fd}{n}\right)^2} \times h$$

$$= \sqrt{\frac{286}{160} - \left(\frac{18}{160}\right)^2} \times 4 = \sqrt{1.7748} = 1.33 \times 4$$

$$\approx 5.32$$

Here, approximate mean A = 10.5

$n = 160$, $h = 4$,

$\sum fd = 18$ and $\sum fd^2 = 286$

Individual task:

The mass number of the weekly wages (in hundreds of taka) of factory workers is given.

Weekly wages	20 – 22	23 – 25	26 – 28	29 – 31	32 – 34	35 – 37	38 – 40
Number of workers	5	10	26	30	16	8	5

a) How much do the workers of that factory get paid on average per week?

b) Determining the standard deviation of data by the method of assumed mean

Individual task:

Suppose, the average yearly electricity consumption (unit) in your household is as follows:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Unit	322	335	370	883	985	452	402	380	362	350	340	335

a) represent the data on a number line.

b) Determine the arithmetic mean and median of the data.

c) Determine the mean and standard deviation of the data from the arithmetic mean and median.

d) In which two months has the most electricity been used? What could be the reason for this? How many units will your electricity cost be on average if you exclude the electricity cost of those two months? In that case, how much will the standard deviation be?

e) What measures do you think will give you the maximum benefit in terms of electricity consumption throughout the year?

Exercise

1) Determine the range of the following data.

a) 14, 3, 19, 17, 4, 9, 16, 19, 22, 15, 18, 17, 12, 8, 16, 11, 3, 11, 0, 15

b) 48, 70, 58, 40, 43, 55, 63, 46, 56, 44

c)

Height (cm)	95 – 105	105 – 115	115 – 125	125 – 135	135 – 145	145 – 155
Frequency	8	12	28	30	15	7

2) Determine the mean deviation from the arithmetical mean and median of the following data.

a) 8, 15, 53, 49, 19, 62, 7, 15, 95, 77

b) 10, 15, 54, 59, 19, 62, 98, 8, 25, 95, 77, 46, 36

3) Determine the mean deviation from the arithmetical mean and median of the given data.

x	60	61	62	63	64	65	66	67
f	2	0	15	30	25	12	11	5

4) Sobuj and Mouli spend Tk 50 and Tk 80 respectively for going to school by rickshaw every day.

a) Determine the standard deviation of Sobuj and Mouli.

b) Show that the mean of the two data is half the range.

5) In the following is given the data on the number of patients who visit the Out Department of Thana Health Centre for any given day:

Age	0 – 15	15 – 30	30 – 45	45 – 60	60 – 75	75 – 90
Number of patients	15	4	5	9	7	10

a) When is the value of variance minimum? Explain

b) Determine the mean deviation and standard deviation of the data and then compare.

6) 33.2 is the arithmetic mean deviation of the mass number in the following table. Find the value of p by finding the arithmetic mean.

Class range	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
Mass number	8	12	$\boxed{p}$	30	15	10	5

7) Nipa has a flower garden. The garden has 60 different varieties of flowering plants. The mean height (in cm) of the plants is 28.5.

Height (cm)	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Number of trees	5	$\boxed{x}$	20	15	$\boxed{y}$	5

a) Find the values of x and y and complete the table.

b) Calculate the mean height of the trees in a simplified manner.

c) Determine the mean deviation of the height of the trees from the median.

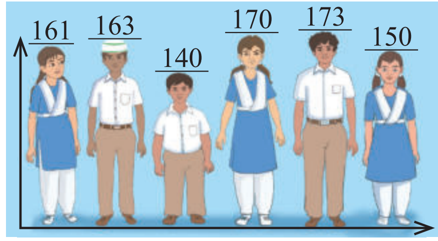
d) Determine the standard deviation of the height of the trees from the mean.

8) See the picture given beside. In the picture, the height of the six students is given in centimetres. Do the following exercises:

a) Determine the mean and median of Height of students

b) Determine the mean deviation from the mean and median of Height of students

c) Determine the standard deviation from the mean and median.

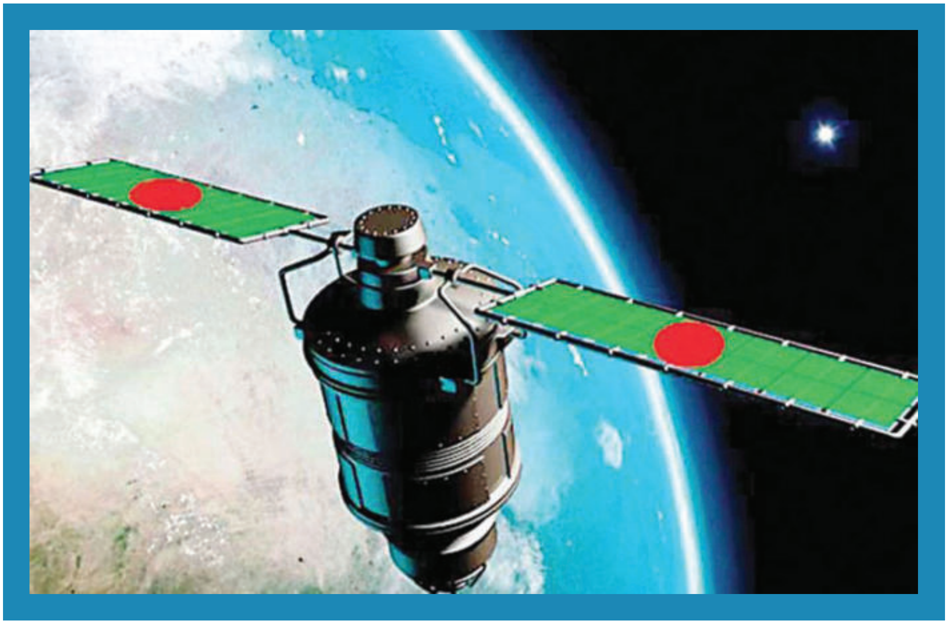


9) The arithmetic mean and standard deviation of a sample of ten members are 9.5 and 2.5 respectively. Later another member of 15 values was included in the sample. Now, find the arithmetic mean and standard deviation of a sample of eleven members.

10) The annual profit (in crores) of 100 companies is given below:

Profit	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Number of companies	7	12	22	30	20	9

Calculate the mean deviation and standard deviation from the arithmetic mean of the data.



বঙ্গবন্ধু স্যাটেলাইট-১ : বাংলাদেশের মালিকানাধীন প্রথম কৃত্রিম উপগ্রহ

বঙ্গবন্ধু স্যাটেলাইট-১ বাংলাদেশের প্রথম ভূস্থির (Geostationary) যোগাযোগ ও সম্প্রচার উপগ্রহ। এর মধ্য দিয়ে ৫৭ তম দেশ হিসেবে নিজস্ব স্যাটেলাইট উৎক্ষেপণকারী দেশের তালিকায় যুক্ত হয় বাংলাদেশ। এটি ১১ই মে ২০১৮ যুক্তরাষ্ট্রের কেনেডি স্পেস সেন্টার থেকে উৎক্ষেপণ করা হয়। এটি ছিল ফ্যালকন ৯ ব্লক-৫ রকেটের প্রথম পেলোড উৎক্ষেপণ।

এটি ফ্রান্সের থেলিস অ্যালেনিয়া স্পেস কর্তৃক নকশা ও তৈরি করা হয়েছে। বঙ্গবন্ধু স্যাটেলাইট-১, ১৬০০ মেগাহার্টজ ক্ষমতাসম্পন্ন মোট ৪০টি কে-ইউ এবং সি-ব্যান্ড ট্রান্সপন্ডার বহন করছে এবং এর আয়ু ১৫ বছর। এর নির্মাণ ব্যয় প্রায় তিন হাজার কোটি টাকা। বর্তমানে স্যাটেলাইটের ব্যান্ডউইথ ও ফ্রিকোয়েন্সি ব্যবহার করে ইন্টারনেট বঞ্চিত অঞ্চল যেমন-পার্বত্য ও হাওড় এলাকায় ইন্টারনেট সুবিধা প্রদান করা সম্ভব হচ্ছে, প্রত্যন্ত অঞ্চলে ইন্টারনেট ও ব্যাংকিং সেবা, টেলিমেডিসিন ও দূরশিক্ষণ ব্যবস্থা প্রসারেও এটি ব্যবহৃত হচ্ছে। টিভি চ্যানেলগুলো তাদের সম্প্রচার সঠিকভাবে পরিচালনার জন্য বিদেশি নির্ভরতা কমিয়ে এর উপর নির্ভর করছে। ফলে দেশের টাকা দেশেই থাকছে। বড় প্রাকৃতিক দুর্যোগের সময় মোবাইল নেটওয়ার্ক অচল হয়ে পড়লে এর মাধ্যমে দুর্গত এলাকায় যোগাযোগ চালু রাখা সম্ভব। শুধু তাই নয় বঙ্গবন্ধু স্যাটেলাইট-২ মহাকাশে উৎক্ষেপণেরও উদ্যোগ নেওয়া হয়েছে। বঙ্গবন্ধু ১৯৭৫ সালের ১৪ই জুন বেতবুনিয়ায় ভূ-উপগ্রহ কেন্দ্র স্থাপনের মাধ্যমে যে স্বপ্নের বীজ বপন করেছিলেন, সেই স্বপ্ন মহীরূহে পরিণত করেছেন প্রধানমন্ত্রী শেখ হাসিনা।

স্যাটেলাইটের বাইরের অংশে বাংলাদেশের লাল-সবুজ পতাকার রঙের নকশার উপর ইংরেজিতে লেখা রয়েছে বাংলাদেশ ও বঙ্গবন্ধু-১, বাংলাদেশ সরকারের একটি মনোগ্রামও সেখানে রয়েছে।

Academic Year 2024
Class Nine
Mathematics

'একজন ঘুমন্ত মানুষ আরেকজন ঘুমন্ত মানুষকে জাগিয়ে তুলতে পারে না।'
–শেখ সাদি

সমৃদ্ধ বাংলাদেশ গড়ে তোলার জন্য যোগ্যতা অর্জন করো
– মাননীয় প্রধানমন্ত্রী শেখ হাসিনা

তথ্য, সেবা ও সামাজিক সমস্যা প্রতিকারের জন্য **'৩৩৩'** কলসেন্টারে ফোন করুন

নারী ও শিশু নির্যাতনের ঘটনা ঘটলে প্রতিকার ও প্রতিরোধের জন্য ন্যাশনাল হেল্পলাইন সেন্টার
১০৯ নম্বর-এ (টোল ফ্রি, ২৪ ঘণ্টা সার্ভিস) ফোন করুন

শিক্ষা নিয়ে গড়ব দেশ
শেখ হাসিনার বাংলাদেশ

Ministry of Education

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